11. Eigenvalues and eigenvectors

We have seen in the last chapter:
for the centroaffine mapping

$$f: \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \to \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} \cdot \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} \lambda_1 x_1 \\ \lambda_2 x_2 \end{bmatrix},$$

some directions, namely, the directions of the coordinate axes: $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$, are distinguished among all directions in the plane: In them, f acts as a *pure scaling*.
We want to generalize this to arbitrary linear mappings.

We call a vector representing such a direction an *eigenvector* of the linear mapping f (or of the corresponding matrix A), and the scaling factor which describes the effect of f on it an *eigenvalue*.

Examples:
$\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ is eigenvector of the matrix $\begin{bmatrix} 3 & 0 \\ 0 & 7 \end{bmatrix}$ to the eigenvalue 3:

$$\begin{bmatrix} 3 & 0 \\ 0 & 7 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 3 \\ 0 \end{bmatrix} = 3 \cdot \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$\begin{bmatrix} 2 \\ 0 \end{bmatrix}$ is also eigenvector of $\begin{bmatrix} 3 & 0 \\ 0 & 7 \end{bmatrix}$ to the eigenvalue 3:

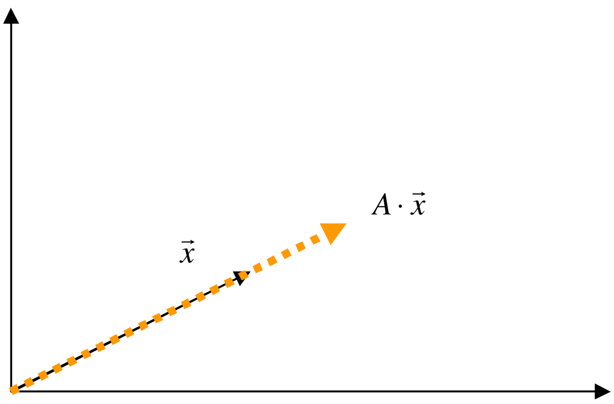
$$\begin{bmatrix} 3 & 0 \\ 0 & 7 \end{bmatrix} \cdot \begin{bmatrix} 2 \\ 0 \end{bmatrix} = \begin{bmatrix} 6 \\ 0 \end{bmatrix} = 3 \cdot \begin{bmatrix} 2 \\ 0 \end{bmatrix}$$

$\begin{bmatrix} 0 \\ 1 \end{bmatrix}$ is eigenvector of $\begin{bmatrix} 3 & 0 \\ 0 & 7 \end{bmatrix}$ to the eigenvalue 7:

$$\begin{bmatrix} 3 & 0 \\ 0 & 7 \end{bmatrix} \cdot \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 7 \end{bmatrix} = 7 \cdot \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

in general:
An eigenvector of *A* must fulfill $A \cdot \vec{x} = \lambda \cdot \vec{x}$, and we require $\vec{x} \neq \vec{0}$.

Definition:
Let *A* be a matrix of type (*n*, *n*). If there exists a real number λ such that the equation $A \cdot \vec{x} = \lambda \cdot \vec{x}$ has a solution $\vec{x}_\lambda \neq \vec{0}$, we call λ an *eigenvalue* and $\vec{x}_\lambda$ an *eigenvector* of the matrix *A*.



If $\vec{x}_\lambda$ is an eigenvector of *A* and $a \neq 0$ an arbitrary factor, then also $a \cdot \vec{x}_\lambda$ is an eigenvector of *A*.
We can choose a in a way that the length of $a \cdot \vec{x}_\lambda$ becomes 1. That means, we can always find eigenvectors of length 1.

If we insert $\vec{x} = E\,\vec{x}$, we can transform the equation $A \cdot \vec{x} = \lambda \cdot \vec{x}$ in the following way:

$$
\begin{aligned}
A \cdot \vec{x} = \lambda \cdot \vec{x} \quad &\Leftrightarrow \quad A \cdot \vec{x} - \lambda \cdot \vec{x} = \vec{0} \\
&\Leftrightarrow \quad A\,\vec{x} - \lambda\,E\,\vec{x} = \vec{0} \\
&\Leftrightarrow \quad (A - \lambda\,E)\,\vec{x} = \vec{0}
\end{aligned}
$$

This is equivalent to a system of linear equations with matrix $A - \lambda E$ and with right-hand side always zero.
If the matrix $A - \lambda E$ has maximal rank (i.e., if it is regular), this system has exactly one solution (i.e., the trivial solution: the zero vector). We are *not* interested in *that* solution!
The system has other solutions (infinitely many ones), if and only if $A - \lambda E$ is singular,
that means, if and only if

$$\det(A - \lambda E) = 0.$$

From this, we can derive a method to determine all eigenvalues and eigenvectors of a given matrix.

The equation $\det(A - \lambda E) = 0$ (called the *characteristic equation* of A) is an equation between numbers (not vectors) and includes the unknown λ. Solving it for λ means finding all possible eigenvalues of A.

In the case of a 2×2 matrix A, the characteristic equation $\det(A - \lambda E) = 0$ has the form

$$\begin{vmatrix} a_{11}-\lambda & a_{12} \\ a_{21} & a_{22}-\lambda \end{vmatrix} = \lambda^2-(a_{11}+a_{22})\lambda+a_{11}a_{22}-a_{12}\cdot a_{21}=0$$

i.e., it is a quadratic equation and can be solved with the well-known pq formula (see Chapter 6, p. 28).

Example:

$$A = \begin{bmatrix} 1 & -\frac{1}{2} \\ -\frac{1}{2} & 1 \end{bmatrix}$$

$$A-\lambda E = \begin{bmatrix} 1 & -\frac{1}{2} \\ -\frac{1}{2} & 1 \end{bmatrix} - \lambda \cdot \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1-\lambda & -\frac{1}{2} \\ -\frac{1}{2} & 1-\lambda \end{bmatrix}$$

$$\det(A-\lambda E) = \begin{vmatrix} 1-\lambda & -\frac{1}{2} \\ -\frac{1}{2} & 1-\lambda \end{vmatrix} = (1-\lambda)^2-\left(-\frac{1}{2}\right)^2 = 1-2\lambda+\lambda^2-\frac{1}{4}$$

$$= \lambda^2-2\lambda+\frac{3}{4}\overset{!}{=}0$$

$$\Leftrightarrow \lambda_{1,2}=1\pm\sqrt{1-\frac{3}{4}}=1\pm\frac{1}{2}, \quad \lambda_1=\frac{1}{2}, \quad \lambda_2=\frac{3}{2}$$

$\lambda^2-2\lambda+\frac{3}{4}$ is called the *characteristic polynomial* of A.

Its zeros, the solutions $\lambda_1=\frac{1}{2}, \quad \lambda_2=\frac{3}{2}$, are the eigenvalues of A.

That means: Exactly for $\lambda = \frac{1}{2}$ and $\lambda = \frac{3}{2}$ does the vector equation $A \cdot \vec{x} = \lambda \cdot \vec{x}$ have nontrivial solution vectors $\vec{x} \neq \vec{0}$, i.e., eigenvectors.

The next step is to find these eigenvectors vor each of the eigenvalues:

This means to solve a system of linear equations!

We use the equivalent form $(A - \lambda E)\vec{x} = \vec{0}$.
We are not interested in the trivial solution $\vec{x} = \vec{0}$.

In the example: To find an eigenvector to the eigenvalue $\lambda_1 = \frac{1}{2}$: $(A - \frac{1}{2}E)\vec{x} = \vec{0}$

$$\Leftrightarrow \begin{bmatrix} 1-\frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{2} & 1-\frac{1}{2} \end{bmatrix} \cdot \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\Leftrightarrow \begin{bmatrix} \frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} \end{bmatrix} \cdot \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

(system of 2 linear equations with r.h.s. 0) with elementary row operations we get:

$\frac{1}{2}$	$-\frac{1}{2}$	⋮	0
$-\frac{1}{2}$	$\frac{1}{2}$	⋮	0
1	−1	⋮	0
1	−1	⋮	0
1	−1	⋮	0
0	0	⋮	0

From the second-last row we deduce:

$x_1 + (-x_2) = 0$

We can choose one parameter arbitrarily, e.g., $x_2 = c$, and obtain the general solution

$\vec{x} = \begin{bmatrix} c \\ c \end{bmatrix}$ (with $c \in \mathbb{R}$ and $c \neq 0$ because we want to have an eigenvector)

It is enough to give just one vector as a representative of this direction, e.g.,

$\vec{x} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

This is an eigenvector of *A* to the eigenvalue 1/2.

Test: $A \cdot \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & -\frac{1}{2} \\ -\frac{1}{2} & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} \end{bmatrix} = \frac{1}{2} \cdot \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \lambda_1 \cdot \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

The eigenvectors to the second eigenvalue, 3/2, are determined analogously
(a solution is $\begin{pmatrix} -1 \\ 1 \end{pmatrix}$.)

In the general case of an $n \times n$ matrix, $\det(A - \lambda E)$ is a *polynomial* in the variable λ of degree n, i.e., when we develop the determinant, we get something of the form

$$c_n \lambda^n + c_{n-1} \lambda^{n-1} + \ldots + c_1 \lambda + c_0$$

Such a polynomial has at most n zeros, so A can have at most n different eigenvalues.

Attention:
There are matrices which have no (real) eigenvalues at all!
Example: Rotation matrices with angle $\varphi \neq 0°, 180°$.

It is also possible that for the same eigenvalue, there are different eigenvectors with different directions.

Example: For the scaling matrix $A = \begin{bmatrix} 5 & 0 \\ 0 & 5 \end{bmatrix}$, every vector $\vec{x} \neq \vec{0}$ is eigenvector to the eigenvalue 5.

Fixed points and attractors

Let f: $\mathrm{IR}^n \rightarrow \mathrm{IR}^n$ be an arbitrary mapping.
$\vec{x} \in \mathrm{IR}^n$ is called a *fixed point* of f, if $f(\vec{x}) = \vec{x}$, i.e., if $\vec{x}$ remains "fixed" under the mapping f.

$\vec{x}$ is called *attracting fixed point*, *point attractor* or *vortex point* of f, if there exists additionally a neighbourhood of $\vec{x}$ such that for each $\vec{y}$ from this neighbourhood the sequence

$$\vec{y},\ f(\vec{y}),\ f(f(\vec{y})),\ \ldots$$

converges against $\vec{x}$.

The fixed points of linear mappings are exactly (by definition) the eigenvectors to the eigenvalue 1 and the zero vector.

Examples:

$A = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$ (shear mapping): each point on the *x* axis is a fixed point.

$A = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$ (scaling by 2): only the origin (0; 0) is fixed point. (There are no eigenvectors to the eigenvalue 1; the only eigenvalue is 2.)
The origin is not attracting.

$A = \begin{bmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{bmatrix}$ (scaling by 1/2, i.e., shrinking): the origin (0; 0) is attracting fixed point.

Definition:
A stochastic matrix is an $n \times n$ matrix where all columns sum up to 1.

Theorem:
Each stochastic matrix has the eigenvalue 1.
The corresponding linear mapping has thus a fixed point $\neq \vec{0}$.

Example from epidemiology:

The outbreak of a disease is conceived as a stochastic (random) process. For a tree there are two possible states:

"healthy" (state 0) and
"sick" (state 1).

For a healthy tree, let us assume a probability of 1/4 to be sick after one year, i.e.:
$p_{01} = \frac{1}{4}$, and correspondingly: $p_{00} = \frac{3}{4}$ (= probability to stay healthy).
For sick trees, we assume a probability of spontaneous recovery of 1/3:
$p_{10} = \frac{1}{3}$, $p_{11} = \frac{2}{3}$
We define the *transition matrix* (similar to the age-classes example) as

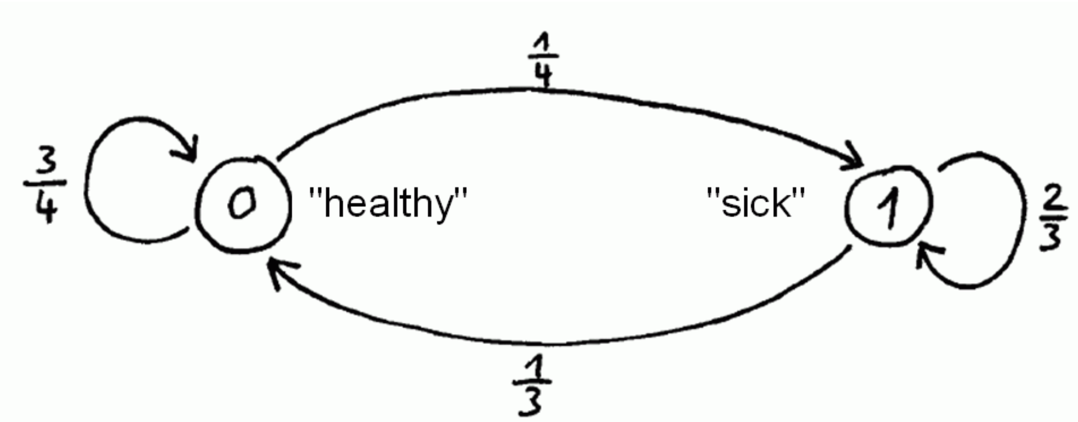
$$P = \begin{bmatrix} p_{00} & p_{01} \\ p_{10} & p_{11} \end{bmatrix}$$

For the purpose of calculation, we need the transposed of *P*, which is a stochastic matrix (and is in the literature also often called the transition matrix):

$$P^T = \begin{bmatrix} \frac{3}{4} & \frac{1}{3} \\ \frac{1}{4} & \frac{2}{3} \end{bmatrix}$$

A process of this sort, where the probability to come into a new state depends only on the current state, is called a *Markov chain*.

Graphical representation of the transitions:



If we assume that g_1, resp., k_1 are the proportions of healthy, resp., sick trees in the first year, the average proportions in the 2nd year are given by:

$$\begin{bmatrix} g_2 \\ k_2 \end{bmatrix} = P^T \cdot \begin{bmatrix} g_1 \\ k_1 \end{bmatrix} = \begin{bmatrix} \frac{3}{4} \cdot g_1 + \frac{1}{3} \cdot k_1 \\ \frac{1}{4} \cdot g_1 + \frac{2}{3} \cdot k_1 \end{bmatrix}$$

Question: what is the percentage of sick trees, if the tree stand is undisturbed for many years and the transition probabilities remain the same?

We have to look for a fixed point of the mapping corresponding to P^T.
Because P^T is a stochastic matrix, it has automatically the eigenvalue 1.
We have only to determine a corresponding eigenvector (fixed point) $\begin{pmatrix} g' \\ k' \end{pmatrix}$:

$$\begin{bmatrix} \frac{3}{4}-1 & \frac{1}{3} \\ \frac{1}{4} & \frac{2}{3}-1 \end{bmatrix} \cdot \begin{bmatrix} g' \\ k' \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

By applying the standard method for solving linear systems, we obtain:

$$\begin{bmatrix} g' \\ k' \end{bmatrix} = \begin{bmatrix} 4\cdot c \\ 3\cdot c \end{bmatrix} \quad , c \neq 0$$

.
From this we derive the proportion of the sick trees:

$$\frac{k}{g+k} = \frac{3}{4+3} = \frac{3}{7}$$

Remarks:
This proportion does not depend on the number of sick trees in the first year.
$\begin{pmatrix} g' \\ k' \end{pmatrix}$ is in fact an attracting fixed point, if we restrict ourselves to a fixed total number of trees, $g+k$.

In the same way, a *stable age-class distribution* can be calculated in the case of the age-class transition matrix (see Chapter 10, p. 82-83).

In that case, the stable age-class vector $\vec{a}^*$ has to be determined as the fixed point (eigenvector to the eigenvalue 1) of the matrix P^T, i.e., as the solution to

$$P^T \cdot \vec{a}^* = \vec{a}^*.$$

Because the fixed point is attracting, it can be obtained as the limit of the sequence

$$\vec{a}_0 \ , \ P^T \cdot \vec{a}_0 \ , \ (P^T)^2 \cdot \vec{a}_0 \ , \ (P^T)^3 \cdot \vec{a}_0 \ , \ \ldots ,$$

starting from an initial vector $\vec{a}_0$.