9. The determinant

The *determinant* is a function (with real numbers as values) which is defined for quadratic matrices.

It allows to make conclusions about the rank and appears in diverse theorems and formulas.

Notation:

$\begin{bmatrix} \cdots \\ \cdots \end{bmatrix}$ matrix, $\begin{vmatrix} \cdots \\ \cdots \end{vmatrix}$ determinant.

Also: A matrix, $\det(A) = |A| \in \mathbb{R}$ determinant of A.

$$det\, A = |A| = \begin{vmatrix} a_{11} a_{12} \cdots a_{1n} \\ \cdots\cdots\cdots\cdots \\ \cdots\cdots\cdots\cdots \\ a_{n1} a_{n2} \cdots a_{nn} \end{vmatrix}$$

We call this a determinant of *order* n.

Calculation in the special cases $n = 2$ and $n = 3$:

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$$

$$\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = aei + bfg + cdh - afh - bdi - ceg$$

The formula for the case 3×3 is called "Sarrus' rule".
Other notation for it, using auxiliary columns for better memorizing the products and their signs:

$$|A| = \left| \begin{array}{ccc|cc} a_{11} & a_{12} & a_{13} & a_{11} & a_{12} \\ a_{21} & a_{22} & a_{23} & a_{21} & a_{22} \\ a_{31} & a_{32} & a_{33} & a_{31} & a_{32} \end{array} \right.$$

$$= a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} - a_{13}a_{22}a_{31} - a_{11}a_{23}a_{32} - a_{12}a_{21}a_{33}$$

The calculation formula for the general case requires the notion of a *subdeterminant*:

Let A be an $n \times n$ matrix. Its determinant is $|A|$.
By omitting the ith row and the jth column we obtain a subdeterminant of order $n-1$.
Notation: $|A_{ij}|$.

In the following formulas, the value of this subdeterminant is multiplied by the factor $(-1)^{i+j}$, giving a sign which alternates between rows and columns like in a chessboard:

$$\begin{vmatrix} + & - & + & - & \cdots \\ - & + & - & + & \\ + & - & + & - & \\ - & + & - & + & \cdots \\ \vdots & & & \vdots & \end{vmatrix}$$

Formulas for calculating determinants of $n \times n$ matrices A of arbitrary size (so-called *development theorems*):

(a) For a fixed jth column:

$$|A| = \sum_{i=1}^{n} (-1)^{i+j} a_{ij} |A_{ij}|$$

(b) For a fixed ith row:

$$|A| = \sum_{j=1}^{n} (-1)^{i+j} a_{ij} |A_{ij}|$$

On the right-hand side we have again determinants, but with smaller size.

We call this "to develop a determinant for a given column (or row)".

Example $n = 3$:

$$|A| = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

$$= a_{11} \cdot (-1)^{1+1} \cdot \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} + a_{12} \cdot (-1)^{1+2} \cdot \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13} \cdot (-1)^{1+3} \cdot \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$$

When we have zero entries, it is advantageous to choose the rows or columns with most zeros.

Example for $n = 4$:

$$|A| = \begin{vmatrix} 1 & 2 & 1 & 2 \\ 0 & 1 & 8 & 7 \\ 0 & 3 & 3 & 1 \\ 0 & 1 & 0 & 1 \end{vmatrix} = (-1)^{1+1} \cdot \begin{vmatrix} 1 & 8 & 7 \\ 3 & 3 & 1 \\ 1 & 0 & 1 \end{vmatrix} = 3+8+0-21-0-24 = -34$$

Rules for determinants:

(1) Switching two rows or two columns changes the sign of the determinant.
(2) If a matrix has a zero row (or a zero column), its determinant is 0.
(3) Has a matrix two identical rows (or columns), its determinant is 0.
(4) If a row (column) of a matrix is multiplied by k, the value of its determinant increases also by the factor k.
(5) If some row (column) is a linear combination of the other rows (columns), the determinant is 0.
(6) The determinant does not change its value if some linear combination of the other rows (columns) is added to a row (column).
(7) The determinant of a matrix does not change if the matrix is transposed: $|A| = |A^T|$.
(8) The determinant of a triangular matrix is the product of the elements of the principal diagonal.

Definition "regular" / "singular":
An $n \times n$ matrix A is called *regular* if $rank(A) = n$ (i.e., if it has the maximal possible rank)
– or, expressed in another way: if all its rows (columns) are linearly independent

Otherwise, A is called *singular*.

Theorem: A is regular $\Leftrightarrow$ $|A| \neq 0$.

Corollary: A is singular $\Leftrightarrow$ $|A| = 0$.

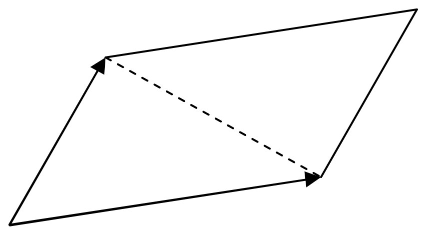
We can thus use the determinant as an indicator of linear independence (of all rows or columns of *A*).

Geometrical application of the determinant:

When the sign is disregarded, $\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}$ is the area content of the *parallelogram* spanned by the two column vectors $\vec{a}=\begin{bmatrix} a_{11} \\ a_{12} \end{bmatrix}$ and $\vec{b}=\begin{bmatrix} a_{21} \\ a_{22} \end{bmatrix}$.

$$\left|det(\vec{a},\vec{b})\right|=\left|a_{11}a_{22}-a_{12}a_{21}\right|$$

(Remark: The area of the spanned *triangle* is exactly half of this value!)



Analogously in IR^3 :
Disregarding the sign, $det(\vec{a},\vec{b},\vec{c})$ is the volume of the *parallelepiped* spanned by $\vec{a},\vec{b},\vec{c}$.

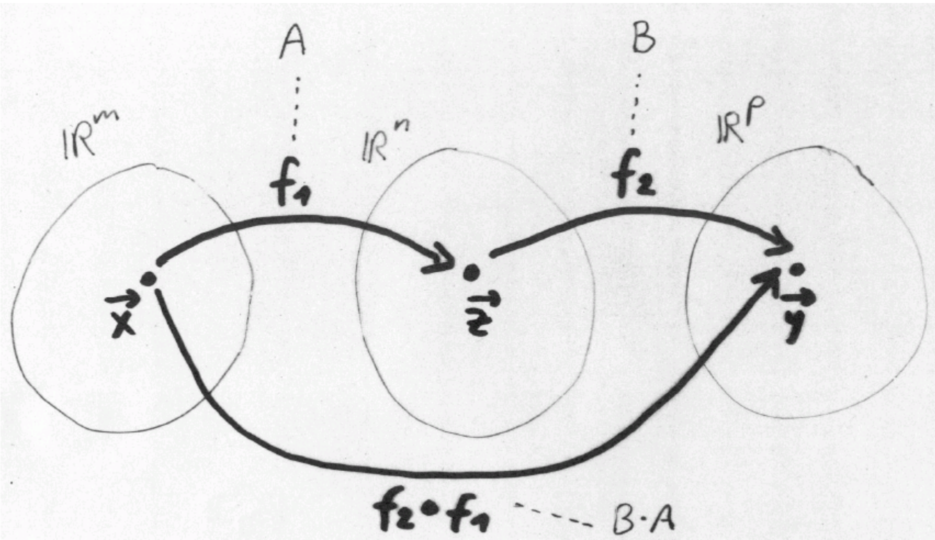
10. More on linear mappings and matrices

Linear mappings can be carried out one after the other (composition of mappings):
Let

$f_1 : \mathbb{R}^m \to \mathbb{R}^n$ be described by the matrix A,

$f_2 : \mathbb{R}^n \to \mathbb{R}^p$ be described by the matrix B.

By composing both mappings, we obtain a new mapping $f_2 \circ f_1$, the *composition* of f_1 and f_2 (notice the notation from right to left):



The new mapping $f_2 \circ f_1 : \mathbb{R}^m \to \mathbb{R}^p$ is again linear and has also a corresponding matrix (of type (p, m)).
Its matrix is called the *product* of the matrices A and B: $B \cdot A$

How is the product of two matrices calculated?

The case of two matrices of type (2, 2):

$$\begin{bmatrix} \boxed{b_{11} \quad b_{12}} \\ b_{21} \quad b_{22} \end{bmatrix} \cdot \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} = \begin{bmatrix} \boxed{b_{11}\cdot a_{11}+b_{12}\cdot a_{21}} & b_{11}\cdot a_{12}+b_{12}\cdot a_{22} \\ b_{21}\cdot a_{11}+b_{22}\cdot a_{21} & b_{21}\cdot a_{12}+b_{22}\cdot a_{22} \end{bmatrix}$$

All possible inner products "row of the first matrix" by "column of the second matrix" are calculated and written into the result matrix.

This holds also in the general case:

Definition:
The *product* of two matrices A of type (m, n) and B of type (n, p) is a matrix $C = A \cdot B$ of type (m, p) with the elements

$$c_{ij} = \sum_{k=1}^{n} a_{ik} b_{kj} .$$

The product exists only in the case when the first matrix has as many columns as the second has rows!

Example:

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 3 & 0 & 2 \\ -1 & -2 & 0 \end{bmatrix} , \quad B = \begin{bmatrix} 2 & -1 \\ 3 & 2 \\ 0 & 1 \end{bmatrix} \quad \Rightarrow$$

$$A \cdot B = \begin{bmatrix} 8 & 6 \\ 6 & -1 \\ -8 & -3 \end{bmatrix}$$

Attention: In the general case, $A \cdot B \neq B \cdot A$.

Example: $\begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} \cdot \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 2 \\ 0 & 0 \end{pmatrix}$, but $\begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$.

Transposition of a matrix product:

$$(A \cdot B)^{\mathrm{T}} = B^{\mathrm{T}} \cdot A^{\mathrm{T}}$$

The product of a matrix A with a column vector $\vec{v}$ is a special case of the product of two matrices (second factor of type (n, 1)).

<u>Application:</u>

Transformation of age-class vectors

We remember:

The age-class structure of a forest at time t can be described by an age-class vector.

$$\begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix} \begin{array}{l} \longleftarrow \text{area with trees of age class 1} \\ \longleftarrow \text{area with trees of age class 2} \\ \\ \longleftarrow \text{area with trees of age class } n \end{array}$$

$= \vec{a}_t \in \mathbb{R}^n$ (n = number of successive age classes)

The development of the age structure over time can be described by a linear mapping $\mathbb{R}^n \to \mathbb{R}^n$.

Let $p_{j,k}$ be the part of the area of the jth age class which comes into the kth age class.

Example: $p_{3,4} = 0.7$
$p_{3,1} = 0.3$, i.e., 30% of the stand of age class 3 are cut and the free area is immediately reforested with young trees (age class 1)
$p_{3,k} = 0$ for all other k.

Age-class transition matrix:

$$P := \begin{bmatrix} p_{1,1} & p_{1,2} & \cdots & p_{1,n} \\ \vdots & & & \vdots \\ p_{n,1} & p_{n,2} & \cdots & p_{n,n} \end{bmatrix}$$

In the calculations, more often the transposed matrix P^T is used. In population ecology, it is called the *Leslie matrix*.

Theorem:
The age class vector of the stand at time t+1 can be calculated as

$$\vec{a_{t+1}} = P^T \cdot \vec{a}_t$$

In the simple case n=3, this gives

$$\begin{bmatrix} p_{1,1} & p_{2,1} & p_{3,1} \\ p_{1,2} & p_{2,2} & p_{3,2} \\ p_{1,3} & p_{2,3} & p_{3,3} \end{bmatrix} \cdot \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} p_{1,1} a_1 + p_{2,1} a_2 + p_{3,1} a_3 \\ p_{1,2} a_1 + p_{2,2} a_2 + p_{3,2} a_3 \\ p_{1,3} a_1 + p_{2,3} a_2 + p_{3,3} a_3 \end{bmatrix}$$

Because some of the entries of P^T are necessarily 0 (organisms cannot stop ageing; they cannot overjump some age class or become younger), this can be simplified to:

$$\begin{bmatrix} p_{1,1} & p_{2,1} & p_{3,1} \\ p_{1,2} & 0 & 0 \\ 0 & p_{2,3} & 0 \end{bmatrix} \cdot \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} p_{1,1} a_1 + p_{2,1} a_2 + p_{3,1} a_3 \\ p_{1,2} a_1 \\ p_{2,3} a_2 \end{bmatrix}$$

We can say that P describes a forest management strategy.

Usage of the matrix product in this context:

If between times t and $t+1$, strategy P is applied, and between times $t+1$ and $t+2$ strategy Q, then we have in total:

$$\vec{a}_{t+2} = Q^T \cdot \vec{a}_{t+1} = Q^T \cdot (P^T \cdot \vec{a}_t) = (Q^T \cdot P^T) \cdot \vec{a}_t = (P \cdot Q)^T \cdot \vec{a}_t$$

If the strategy is *the same* in every time step, we have:

$$\begin{aligned} \vec{a}_t &= P^T \cdot \vec{a}_{t-1} \\ &= P^T \cdot P^T \cdot \vec{a}_{t-2} \\ &= P^T \cdot P^T \cdot P^T \cdot \vec{a}_{t-3} \\ &= \ldots \\ &= (P^T)^t \cdot \vec{a}_0 \end{aligned}$$

(Here, $()^T$ means transposition, $()^t$ means the t-fold product of a matrix with itself.)

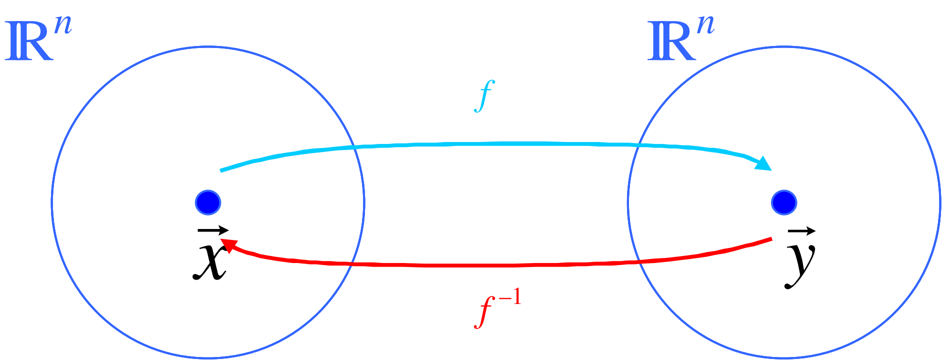
<u>The inverse matrix</u>

Let A be an $n \times n$ matrix.
A^{-1} is called the inverse matrix of A if
$A^{-1} \cdot A = A \cdot A^{-1} = E$ (= the unit matrix).

Not every matrix has an inverse.
If the inverse matrix of A exists, it is unique.
It is always $(A^{-1})^{-1} = A$.

When does A^{-1} exist ?

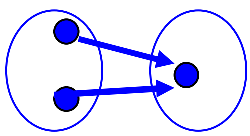
A is a matrix of type (n, n)
corresponding linear mapping $f : \mathbb{R}^n \to \mathbb{R}^n$



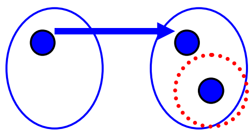
$\vec{y} = f(\vec{x}) = A \cdot \vec{x}$

when does the inverse mapping f^{-1} exist?

If f is bijective, i.e.,
f injective …… not:



and
f surjective ……not:



$\Leftrightarrow f(\vec{e}_1)\ ,\dots, f(\vec{e}_n)$ span $\mathbb{R}^n$ completely

$$\vec{y} = m_1 f(\vec{e}_1) + \dots + m_n f(\vec{e}_n)$$

$$= f(m_1\vec{e}_1 + \dots + m_n\vec{e}_n) = f\begin{pmatrix} m_1 \\ \vdots \\ m_n \end{pmatrix}$$

$$\Leftrightarrow\ rank(A) = n \ \Leftrightarrow\ \det A \neq 0$$

$$\Leftrightarrow\ \underline{\underline{A\ regular}}$$

Exactly the regular $n \times n$ matrices have an inverse matrix.

How to calculate it?
Most efficient way: by elementary row operations.
Concatenate an $n \times n$ unit matrix E to A:

$$M = [\,A \mid E\,] = \left[\begin{array}{cccc|cccc} a_{11} & a_{12} & \dots & a_{1n} & 1 & 0 & \dots & 0 \\ a_{21} & a_{22} & \dots & a_{2n} & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ a_{n1} & a_{n2} & \dots & a_{nn} & 0 & \dots & \dots & 1 \end{array}\right]$$

Then transform this larger matrix by elementary row operations in a way that the left part is transformed into the unit matrix. The resulting right part is then A^{-1}: $[\,E \mid A^{-1}]$.

Example:

$$A=\begin{bmatrix}1 & 0 & 0\\ 3 & 1 & 0\\ 0 & 3 & 1\end{bmatrix}$$

The start scheme is

$$\left[\begin{array}{ccc|ccc}1 & 0 & 0 & 1 & 0 & 0\\ 3 & 1 & 0 & 0 & 1 & 0\\ 0 & 3 & 1 & 0 & 0 & 1\end{array}\right]$$

By subtracting the first row 3 times from the second row, and then the second row 3 times from the third, we get

$$\left[\begin{array}{ccc|ccc}1 & 0 & 0 & 1 & 0 & 0\\ 0 & 1 & 0 & -3 & 1 & 0\\ 0 & 0 & 1 & 9 & -3 & 1\end{array}\right] \Rightarrow \quad A^{-1} = \begin{bmatrix}1 & 0 & 0\\ -3 & 1 & 0\\ 9 & -3 & 1\end{bmatrix}$$

It is recommended to check if really $A \cdot A^{-1} = E$ (otherwise some error must have occurred).

Systems of linear equations

A system of m linear equations with n unknowns can always be ordered and rewritten in the form

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n &= b_2 \\ \vdots \qquad\qquad\qquad & \quad \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \ldots + a_{mn}x_n &= b_m \end{aligned}$$

If we collect the unknowns x_k in a column vector

$\vec{x} \in \mathbb{R}^n$

and the numbers b_i on the right-hand side (called absolute terms) also in a column vector

$\vec{b} \in \mathbb{R}^m$

and the coefficients a_{ik} in a matrix A of type (m, n), we can write the whole system *as a single equation*:

$$A \cdot \vec{x} = \vec{b}$$

$A = \begin{bmatrix} a_{11} & \ldots & a_{1n} \\ a_{21} & \ldots & a_{2n} \\ \vdots & \vdots & \vdots \\ a_{m1} & \ldots & a_{mn} \end{bmatrix}$ is called the *coefficient matrix*

of the system,

$A_{\text{ext}} = \left[\begin{array}{cccc|c} a_{11} & a_{12} & \dots & a_{1n} & b_1 \\ a_{21} & a_{22} & \dots & a_{2n} & b_2 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} & b_m \end{array}\right]$ the *extended matrix* of the system.

Notice:
$A\cdot\vec{x}=\vec{b}$ can also be interpreted as $f(\vec{x})=\vec{b}$, with f the linear mapping described by the matrix *A*.

Finding a solution of the linear system means thus to find a vector which is mapped to $\vec{b}$.

For each system of linear equations, there are three possibilities:
(1) The system has exactly one solution $\vec{x}$,
(2) the system has infinitely many solutions,
(3) the system has no solutions at all (it is then called *inconsistent*).

Examples:
(1) The system $\left\{\begin{matrix} x_1+x_2=5 \\ 2x_1+x_2=7 \end{matrix}\right\}$ has exactly one solution: $\vec{x}=\begin{bmatrix} 2 \\ 3 \end{bmatrix}$, that means, $x_1 = 2$ and $x_2 = 3$.

Indeed, $2 + 3 = 5$ and $2\cdot2 + 3 = 7$, and there are no other combinations of numbers which fulfill both equations simultaneously.

(2) The system $\left\{\begin{matrix} x_1+x_2=5 \\ 2x_1+2x_2=10 \end{matrix}\right\}$ has infinitely many solutions, which all have the form
$\vec{x}=\begin{bmatrix} 5-a \\ a \end{bmatrix}$ ($a \in$ IR), that means, $x_1 = 5-a$ and $x_2 = a$.

(3) The system $\left\{\begin{matrix} x_1+x_2=5 \\ 2x_1+2x_2=7 \end{matrix}\right\}$ has no solution.
Both equations contradict each other.

Frobenius' Theorem:

The system of m linear equations with n unknowns which is described by the vector equation $A\cdot\vec{x}=\vec{b}$ has solutions if and only if $rank(A) = rank(A_{ext})$.
More precisely:
(1) If $rank(A) = rank(A_{ext}) = n$, the system has exactly one solution.
(2) If $rank(A) = rank(A_{ext}) < n$, the system has infinitely many solutions. In this case, the values of $n - rank(A)$ of the unknowns can be chosen arbitrarily.
(3) If $rank(A) \neq rank(A_{ext})$, the system has no solutions at all.

We can check the theorem at the examples from above:

(1) $A=\begin{bmatrix} 1 & 1 \\ 2 & 1 \end{bmatrix}$, $\vec{x}=\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$, $\vec{b}=\begin{bmatrix} 5 \\ 7 \end{bmatrix}$

so, $A_{ext} = \begin{bmatrix} 1 & 1 & 5 \\ 2 & 1 & 7 \end{bmatrix}$, and we have
rank(A) = *rank*(A_{ext}) = 2 = n, there is exactly one solution.

(2) $A = \begin{bmatrix} 1 & 1 \\ 2 & 2 \end{bmatrix}$, $A_{ext} = \begin{bmatrix} 1 & 1 & 5 \\ 2 & 2 & 10 \end{bmatrix}$
Here, *rank*(A) = *rank*(A_{ext}) = 1 < 2 = n (the second row is a multiple of the first one), and we have infinitely many solutions (2–1 = 1 unknown can be put to an arbitrary value).

(3) $A = \begin{bmatrix} 1 & 1 \\ 2 & 2 \end{bmatrix}$, $A_{ext} = \begin{bmatrix} 1 & 1 & 5 \\ 2 & 2 & 7 \end{bmatrix}$
Here, *rank*(A) = 1 < *rank*(A_{ext}) = 2. There is no solution.

How to solve systems of linear equations?

"*Gaussian method of elimination*":
most effective method in the general case.

By *elementary row operations*, the extended matrix of the system is transformed into an upper triangular matrix. The solutions of the corresponding system of equations remain the same!
The system corresponding to the upper triangular matrix can easily be solved "bottom-up" by successive insertion and elimination of unknowns.

Example: Solve the system

$$\begin{aligned} x_1 - 2x_2 + x_3 &= 0 \\ 3x_1 - 5x_2 - 2x_3 &= -3 \\ 7x_1 - 3x_2 + x_3 &= 16 \end{aligned}$$

Its extended matrix is $A_{\text{ext}} = \left[\begin{array}{rrr|r} 1 & -2 & 1 & 0 \\ 3 & -5 & -2 & -3 \\ 7 & -3 & 1 & 16 \end{array}\right]$

By applying elementary row operations, one gets

$$\left[\begin{array}{rrr|r} 1 & -2 & 1 & 0 \\ 0 & 1 & -5 & -3 \\ 0 & 0 & 49 & 49 \end{array}\right]$$ (upper triangular matrix).

From this, we can immediately see that
$rank(A) = rank(A_{\text{ext}}) = 3 = n$,
and following Frobenius we can conclude that the system has exactly one solution.

The system of equations corresponding to the transformed matrix is

$$\begin{aligned} x_1 - 2x_2 + x_3 &= 0 \\ x_2 - 5x_3 &= -3 \\ 49x_3 &= 49 \end{aligned}$$

and this can be solved easily from the third row up to the second and first row by elimination of variables. We obtain:

$x_3 = 1, \quad x_2 = 2, \quad x_1 = 3$

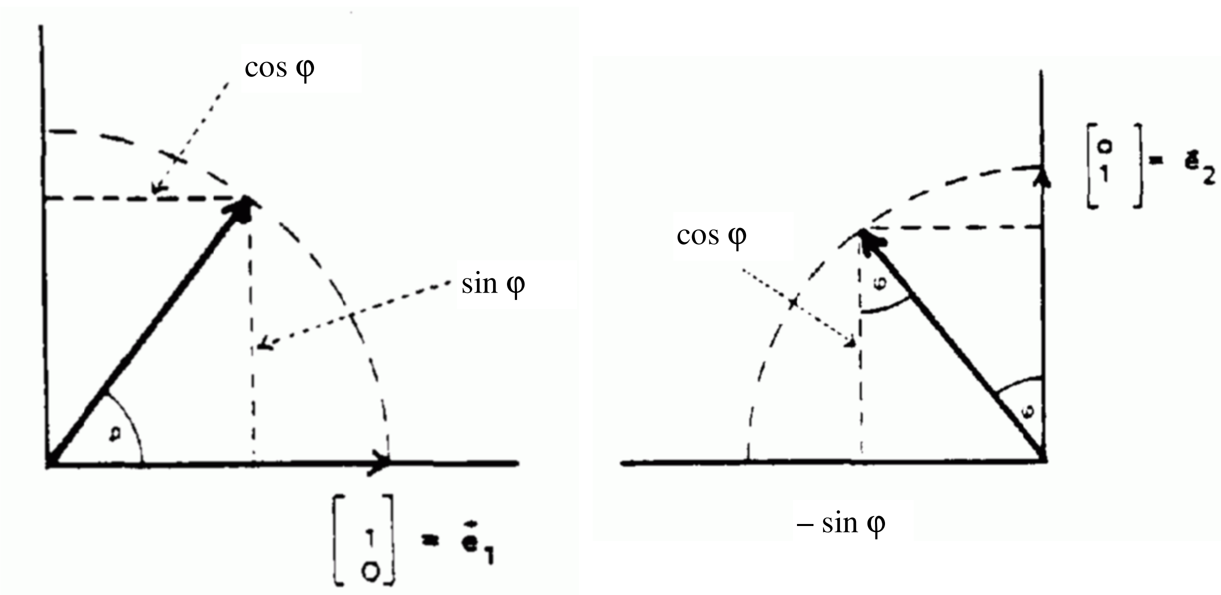
and thus the unique vector solution $\vec{x} = \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}$.

Special cases of linear mappings

(a) Rotations around the origin

Let f be the counterclockwise rotation by the angle φ around the zero point (0; 0) (origin of the cartesian coordinate system).
Each vector is rotated by φ, its image has the same length as before.

Image vectors of the standard basis vectors = ?



We obtain as image vectors: $\begin{pmatrix} \cos\varphi \\ \sin\varphi \end{pmatrix}$ and $\begin{pmatrix} -\sin\varphi \\ \cos\varphi \end{pmatrix}$.

The matrix of f is thus:

$$A = \begin{bmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{bmatrix}$$

We call this a *rotation matrix*.

(b) Scaling

Let f be the mapping which enlarges (or shrinks) every vector by a certain, fixed factor $\lambda \neq 0$.
Its corresponding matrix is

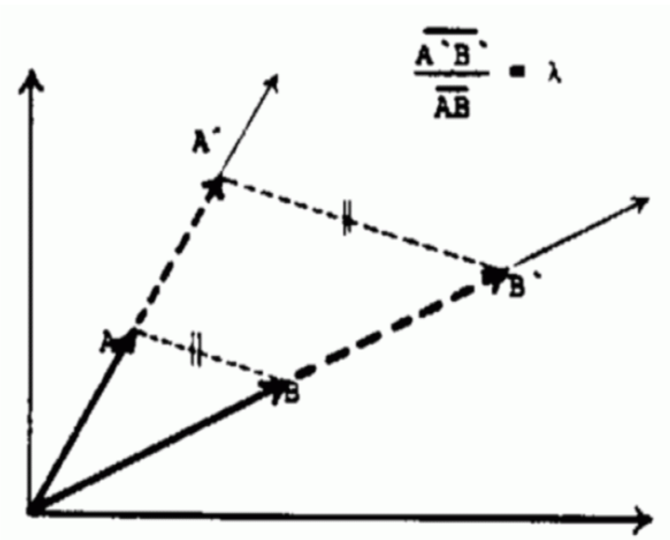
$$A = \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix},$$

called a *scaling matrix* with factor λ.

Indeed, we have

$$A\vec{x} = \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} \cdot \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} \lambda x_1 \\ \lambda x_2 \end{bmatrix}$$

The image of each vector has the same direction as before (or the opposite direction, if λ is negative), but a length which is modified by the factor $|\lambda|$.
Parallelism and all angles remain unchanged under this mapping.



(c) Centroaffine mapping

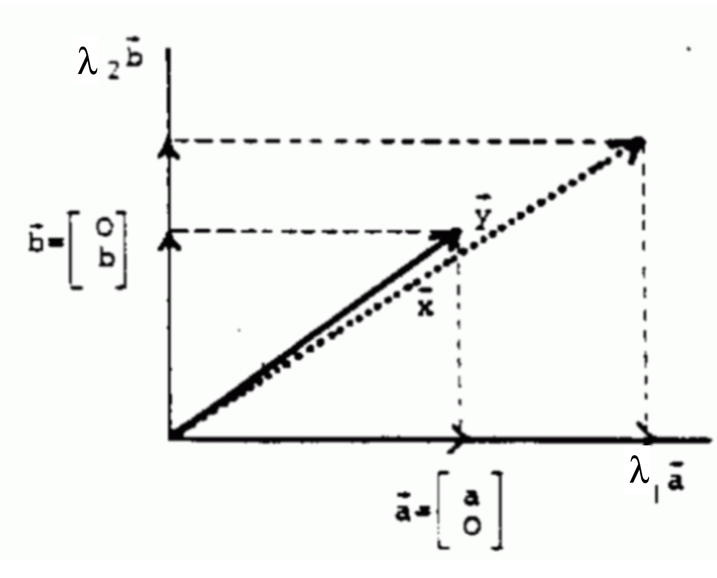
Let f act as a scaling by λ_1 on the *x* axis and as a scaling by λ_2 on the *y* axis, with two different numbers $\lambda_1 \neq \lambda_2$.
The corresponding matrix is

$$A = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}$$

This mapping is called *centroaffine*.

Its effect: $\begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} \cdot \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} \lambda_1 a \\ \lambda_2 b \end{bmatrix}$

It works as a pure scaling on the *x* and *y* axis, but not for vectors which are outside these coordinate axes (they are also rotated a bit):



The centroaffine mapping is thus not a scaling for all vectors.
Certain vectors play a special role for this mapping, namely, those on the coordinate axes: They are only scaled, the others are also rotated.