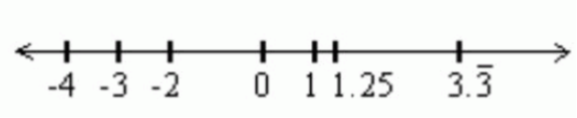
6. Numbers

We do not give an axiomatic definition of the *real numbers* here.

Intuitive meaning: Each point on the (infinite) line of numbers corresponds to a real number, i.e., an element of $\mathbb{R}$.

The line of numbers:



Important subsets of $\mathbb{R}$:

$\mathbb{N}$ the set of all natural numbers (positive integers), does not contain the 0

$\mathbb{N}_0$:= $\mathbb{N} \cup \{0\}$ the set of all non-negative integers

$\mathbb{Z}$ the set of all integers { ... –2; –1; 0; 1; 2; ... }

$\mathbb{Q}$ the set of all rational numbers (representable as fractions of integers p/q, where $q \neq 0$)

We have:

$\mathbb{N} \subseteq \mathbb{Z} \subseteq \mathbb{Q} \subseteq \mathbb{R}$.

Remark:

Every rational number can be represented as decimal number with its expansion after the decimal dot either coming to an end or becoming periodic.

Examples:

$1/4 = 0.25$

$1/7 = 0.\overline{142857}$ (periodic)

$1/6 = 0.1\overline{6}$ (ultimately periodic)

Example for a transformation in the other direction:

$$0,\overline{62} = 0,62 \cdot \left(1 + \frac{1}{100} + \frac{1}{10000} + \frac{1}{1000000} + \ldots\right) = 0,62 \cdot \frac{100}{99} = \frac{62}{99}$$

(note the different notations:
decimal dot in anglosaxon countries, comma in Germany)

Irrational numbers are real numbers that are not rational, i.e., cannot be expressed as a fraction of integers.
Their decimal expansion becomes never periodic.

Examples:

$$\sqrt{2} = 1,41421\ 35623\ 73095\ 04880\ 16887\ 24209\ 69807\ 85696\ 71875\ 37694\ \ldots$$

$$\pi = 3,14159\ 26535\ 89793\ 23846\ 26433\ 83279\ 50288\ 41971\ 69399\ 37510\ldots$$

$$e = 2,71828\,18284\,59045\,23536\,02874\,71352\,66249\,77572\,47093\,69995\ldots$$

Arithmetic operations on IR:

Addition

Operation symbol: +

a + *b* exists for every *a*, *b* ∈ IR.
+ can be seen as a function with two arguments:
a + *b* is in prefix notation +(*a*, *b*).

Rules for adding numbers:
$a + b = b + a$ (*commutativity*)
$(a + b) + c = a + (b + c)$ (*associativity*)
$a + 0 = a$ (0 is the *neutral element* of addition)

For every a, there is a number $-a$ such that
$a + (-a) = 0$

We have always: $-(-a) = a$.

Subtraction can be derived from addition:
$a - b = a + (-b)$.

Multiplication

Operation symbol: $\cdot$ (often omitted!) (sometimes also * instead of $\cdot$).

$a \cdot b$ exists for every $a, b \in \mathbb{R}$.

Rules for multiplication:
$a \cdot b = b \cdot a$
$(a \cdot b) \cdot c = a \cdot (b \cdot c)$
$a \cdot 1 = a$ (1 is the *neutral element* of multiplication)
Rule combining addition and multiplication:
$a \cdot (b + c) = a \cdot b + a \cdot c$ (*distributivity*)

Note: By convention, $\cdot$ binds stronger than +

For every $a \neq 0$, there is a number $1/a$ such that
$a \cdot 1/a = 1$.
We have always: $1/(1/a) = a$.
Other notations for $1/a$: $\frac{1}{a}$, a^{-1}.
$1/a$ is called the *inverse* of a.

Division can be derived from multiplication:
$a : b \; = \; \mathrm{a} \cdot 1/b$

Another notation for $a : b$ is $\frac{a}{b}$.
$a : b$ is not defined for $b = 0$.

The power of a number

A power with a positive integer exponent is defined as an iterated multiplication:
Example: $4^3 = 4 \cdot 4 \cdot 4$.
4 is called the basis, 3 the exponent.

By definition, $a^0 = 1$ for all $a \neq 0$.
For $n > 0$, we define as the power with negative exponent $-n$: $a^{-n} = 1/(a^n)$ ($= (a^n)^{-1}$).

Example: $4^{-3} = (4^3)^{-1} = \frac{1}{4^3} = \frac{1}{64}$

The root of a number

For every positive real number a and every positive integer n there exists a positive real number x which fulfills the equation $x^n = a$.
This (unique) x is called the n-th root of a.
Two notations for x:

$$\sqrt[n]{a} = a^{\frac{1}{n}} \quad \forall\, a \in \mathbb{R}$$

For *odd* integers n and negative a we can extend this definition by $a^{1/n} = -\,(-a)^{1/n}$.

For *even n*, the *n*-th root of a negative number is not defined in $\mathbb{R}$.

To overcome this restriction, it is possible to extend the set of real numbers $\mathbb{R}$:
The so-called imaginary unit $i = \sqrt{-1}$ is defined which fulfills

$$i \cdot i = -1.$$

$\mathbb{R}$ is extended to the set $\mathbb{C}$ of *complex numbers*.
Each complex number has the form $a + b \cdot i$ with $a, b \in \mathbb{R}$.
It is possible to calculate with complex numbers in the same way as with real numbers.
Visualization as points in the plane (with real-valued coordinates *a*, *b*).

Back to the real numbers:

The operation "*n*-th root of..." does invert the power operation.

Attention:

We have (by definition) $(\sqrt{x})^2 = x$,

but: $\sqrt{x^2} = |x|$!

Here, $|x|$ denotes the *absolute value* of *x*:
$|x| = x$ if $x \geq 0$ and $|x| = -x$ otherwise.

$|a - b|$: the *distance* between *a* and *b*.

In the context of square roots, the solution formula for quadratic equations ("*pq formula*") is often a useful tool:

For the equation $x^2 + px + q = 0$,
the solutions (if they exist) are:

$$x_{1,2} = \frac{-p}{2} \pm \sqrt{\frac{p^2}{4} - q}$$

Condition for the existence of the solution(s):

$$\frac{p^2}{4} - q \geq 0$$

For control purposes, Vieta's theorem can be useful:
The two solutions fulfill $x_1 + x_2 = -p$
and $x_1 \cdot x_2 = q$.

The power of real numbers with rational exponent:

The power $a^{k/n}$ is defined as

$$a^{\frac{k}{n}} = \sqrt[n]{a^k}$$
.

(By using limits of series of rational numbers – for the introduction of limits see later – , the definition of a power can also be extended to irrational exponents.)

Rules for powers:

$a^r \cdot a^s = a^{r+s}$
$a^r : a^s = a^{r-s}$
$(a^r)^s = a^{rs}$
$a^r \cdot b^r = (a \cdot b)^r$

Because the power operation a^n is not commutative, there are two different reverse operations: You can search for a basis or you can search for an exponent. The first case leads to the root, the second case to the *logarithm*.

Definition:
Let a, $b > 0$ be real numbers. The (unique) solution of $b^x = a$ is $x = \log_b a$ (logarithm of a to the base b).

Often the so-called *natural logarithm* is used, which uses the Euler number $e = 2.718281828...$ as its base: $\ln a = \log_e a$.
Other frequent cases: binary logarithm (base 2); decimal logarithm (base 10).

In general, we have: $\log_b a = \ln a / \ln b$.

Rules for logarithms (hold for arbitrary base):

$\log(x \cdot y) = \log x + \log y$
$\log (x / y) = \log x - \log y$
$\log (x^y) = y \cdot \log x$

$$\log(\sqrt[n]{x}) = \frac{1}{n} \cdot \log x$$

The order relation on IR

Every two real numbers a, b can be ordered:

Either $a < b$, or $a = b$, or $a > b$.

$a \leq b$ means $a < b$ or $a = b$.

We have:

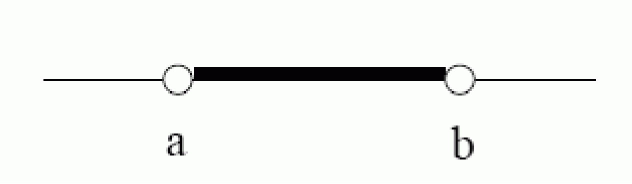
$a < b \Rightarrow a + c < b + c$ (analogously for $\leq$),

for $c > 0$: $a < b \Rightarrow a \cdot c < b \cdot c$

but for $c < 0$: $a < b \Rightarrow a \cdot c > b \cdot c$

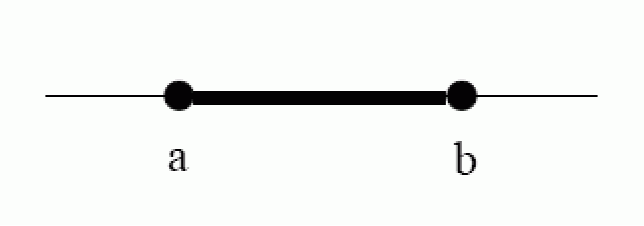
Bounded intervals

An *open, bounded interval* (a, b) is the set of all real numbers x which are *properly between* a and b, i.e., which fulfill $a < x < b$.
Attention! The same notation as for ordered pairs is used, but the meaning is different.
If $a < b$, (a, b) is an infinite set.



(a, b)

In a *closed interval* $[a, b]$, the end points are included: $[a, b] = \{ x \in \mathbb{R} \mid a \leq x \leq b \}$.

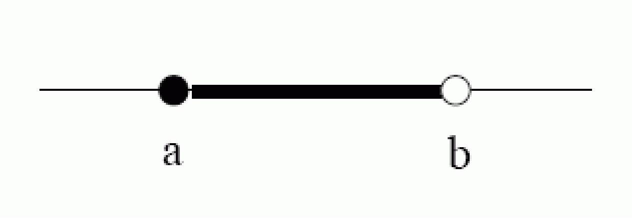


$[a, b]$

An interval closed on the right-hand side:

$(a, b]$

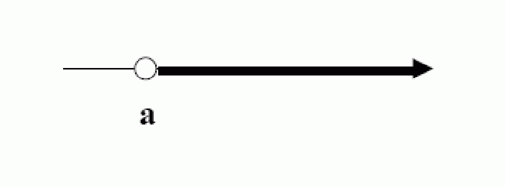
An interval closed on the left-hand side:



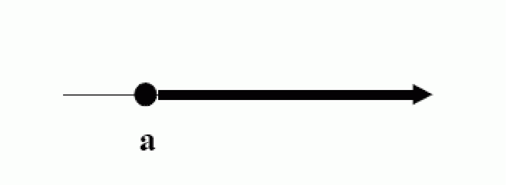
$[a, b)$

Unbounded intervals

$(a, +\infty) = \{ x \in \mathbb{R} \mid a < x \}$.

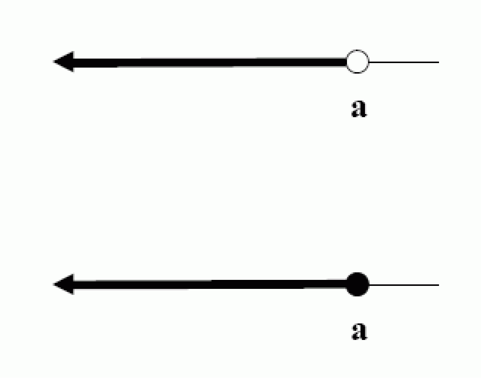


$(a, +\infty)$



$[a, +\infty)$

analogously for intervals unbounded to the left:



$(-\infty, a)$

$(-\infty, a]$

The *neighbourhood* of a number

Let $\varepsilon > 0$ be a positive real number.
The interval $(b - \varepsilon, b + \varepsilon)$ is called the ε-neighbourhood of the number b.

We have $(b - \varepsilon, b + \varepsilon) = \{ x \in \mathrm{IR} \mid |x - b| < \varepsilon \}$.
That means: The neighbourhood contains all numbers for which the *distance* to b is smaller than the given threshold ε.

Bounds

An *upper bound* of a set M of real numbers is a number r with $r > x$ for all $x \in M$.

Analogously: *lower bound* (exchange > by <).

A set of numbers is called *bounded* if there exists an upper bound and a lower bound for it.

If a set has an upper bound, it has infinitely many upper bounds. We are interested in the smallest one:

The smallest upper bound of a set $M \subseteq \mathrm{IR}$ is called the *supremum* of M, denoted sup M.

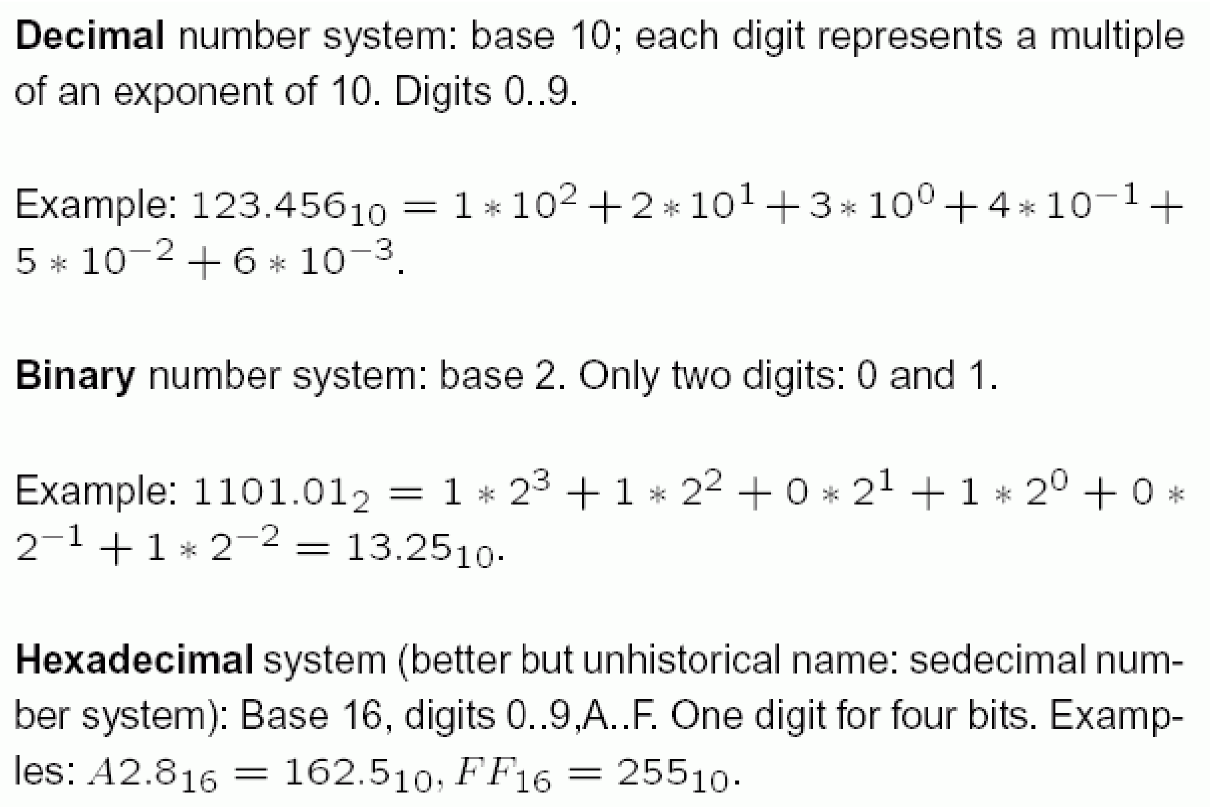
Analogously:
The largest lower bound of a set $M \subseteq \mathrm{IR}$ is called the *infimum* of M, denoted inf M.

Examples:
inf {1; 2; 3; 4} = 1, sup {1; 2; 3; 4} = 4,

$$inf\left\{\frac{1}{n} \;\middle|\; n \in \mathbb{N}\right\} = 0$$

Number systems

Question: How to represent numbers?
We concentrate on positive integers here.



Decimal number system: base 10; each digit represents a multiple of an exponent of 10. Digits 0..9.

Example: $123.456_{10} = 1 * 10^2 + 2 * 10^1 + 3 * 10^0 + 4 * 10^{-1} + 5 * 10^{-2} + 6 * 10^{-3}$.

Binary number system: base 2. Only two digits: 0 and 1.

Example: $1101.01_2 = 1 * 2^3 + 1 * 2^2 + 0 * 2^1 + 1 * 2^0 + 0 * 2^{-1} + 1 * 2^{-2} = 13.25_{10}$.

Hexadecimal system (better but unhistorical name: sedecimal number system): Base 16, digits 0..9,A..F. One digit for four bits. Examples: $A2.8_{16} = 162.5_{10}, FF_{16} = 255_{10}$.

The additional digits in the hexadecimal system:
A = 10, *B* = 11, *C* = 12, *D* = 13, *E* = 14, *F* = 15.

Transformation from one number system to the other:

- Special case (easy): from binary to hexadecimal

Every 4 binary digits correspond directly to a hexadecimal digit

$$\begin{array}{lcccc} \text{Example:} & \underline{0000} & \underline{0010} & \underline{1100} & \underline{0110} \\ \rightarrow & 0 & 2 & C & 6 \end{array}$$

- from arbitrary system to decimal:
  Horner scheme

Input: $z_{n-1}\ z_{n-2}\ \ldots\ z_0$ to base b
start with $h_{n-1} = z_{n-1}$
calculate for $k = n-1, n-2, \ldots, 1$:

$$h_{k-1} = h_k * b + z_{k-1}$$

Output: $z = h_0$

Example:
Input: binary number 1010 ($n = 4$, $b = 2$)
Start: $h_{n-1} = h_3 = z_3 = 1$
$k = n-1 = 3$: $h_2 = h_3 * 2 + z_2 = 1*2 + 0 = 2$
$k = 2$: $h_1 = h_2 * 2 + z_1 = 2*2 + 1 = 5$
$k = 1$: $h_0 = h_1 * 2 + z_0 = 2*5 + 0 = \mathbf{10} = z$

- from decimal to arbitrary:
  Inverse Horner scheme

start with $h_0 = z$ (= input)
calculate for $k = 1, 2, 3, \ldots$:

$$z_{k-1} = h_{k-1} \bmod b,$$
$$h_k = h_{k-1} \operatorname{div} b$$

(mod: rest when dividing by b, div: integral part from dividing by b)

Output: $z_{n-1}\ z_{n-2}\ \ldots\ z_0$ to base b

Example:

Input: decimal number 34, transform in ternary system ($b = 3$)

Start: $h_0 = 34$

$k = 1$: $z_0 = h_0 \bmod 3 = 34 \bmod 3 = 1$,
$h_1 = h_0 \operatorname{div} 3 = 34 \operatorname{div} 3 = 11$

$k = 2$: $z_1 = h_1 \bmod 3 = 11 \bmod 3 = 2$,
$h_2 = h_1 \operatorname{div} 3 = 11 \operatorname{div} 3 = 3$

$k = 3$: $z_2 = h_2 \bmod 3 = 3 \bmod 3 = 0$,
$h_3 = h_2 \operatorname{div} 3 = 3 \operatorname{div} 3 = 1$,

$k = 4$: $z_3 = h_3 \bmod 3 = 1 \bmod 3 = 1$,
$h_4 = h_3 \operatorname{div} 3 = 1 \operatorname{div} 3 = 0$ (Stop)

$\Rightarrow z = 1021$

Remark:

Arbitrary real numbers can also be represented using an arbitrary integer $b > 1$ as base.
Digits after the dot are interpreted as coefficients of b^{-n} ($n = 1, 2, 3, ...$).

Example:

0.111_2 (base b=2) $= 1/2 + 1/4 + 1/8 = 7/8 = 0.875_{10}$

7. Vectors

We will work with elements from the set

$$\mathbb{R}^n = \mathbb{R} \times \mathbb{R} \times \mathbb{R} \times \ldots \times \mathbb{R}$$

The elements are *n*-tuples of real numbers,
we call them *vectors*.

To distinguish vector-valued variables from variables standing for single numbers, often an arrowed letter ($\vec{a}$) or printing in a different font is used.

Two ways to write down a vector:

row vector, e.g., (1; 5; –2)

column vector $\begin{bmatrix} 1 \\ 5 \\ -2 \end{bmatrix}$

To distinguish real numbers from vectors, we call them also *scalars*:

$\vec{a} = \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{pmatrix} \in \mathbb{R}^n$ vector

(for $n = 2; 3$ geometrically:
representation by arrow ; “directed entity“)

$m \in \mathbb{R}$ scalar (“undirected entity“)

$\begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} \cdots$ a_1, a_2, ... are called *components* of the vector

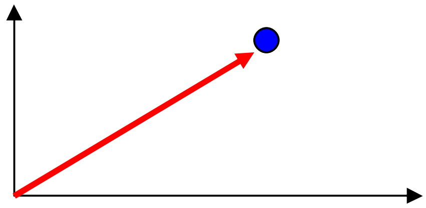
(also: *coordinates*)

special cases:

$\mathrm{IR}^1 = \mathrm{IR}$

$\mathrm{IR}^2 = \mathrm{IR} \times \mathrm{IR}$, can be represented as a plane:

each vector $\begin{bmatrix} x \\ y \end{bmatrix} \in \mathrm{IR}^2$ corresponds to a point in the plane. Often a vector is represented as an arrow pointing from the origin to this point.

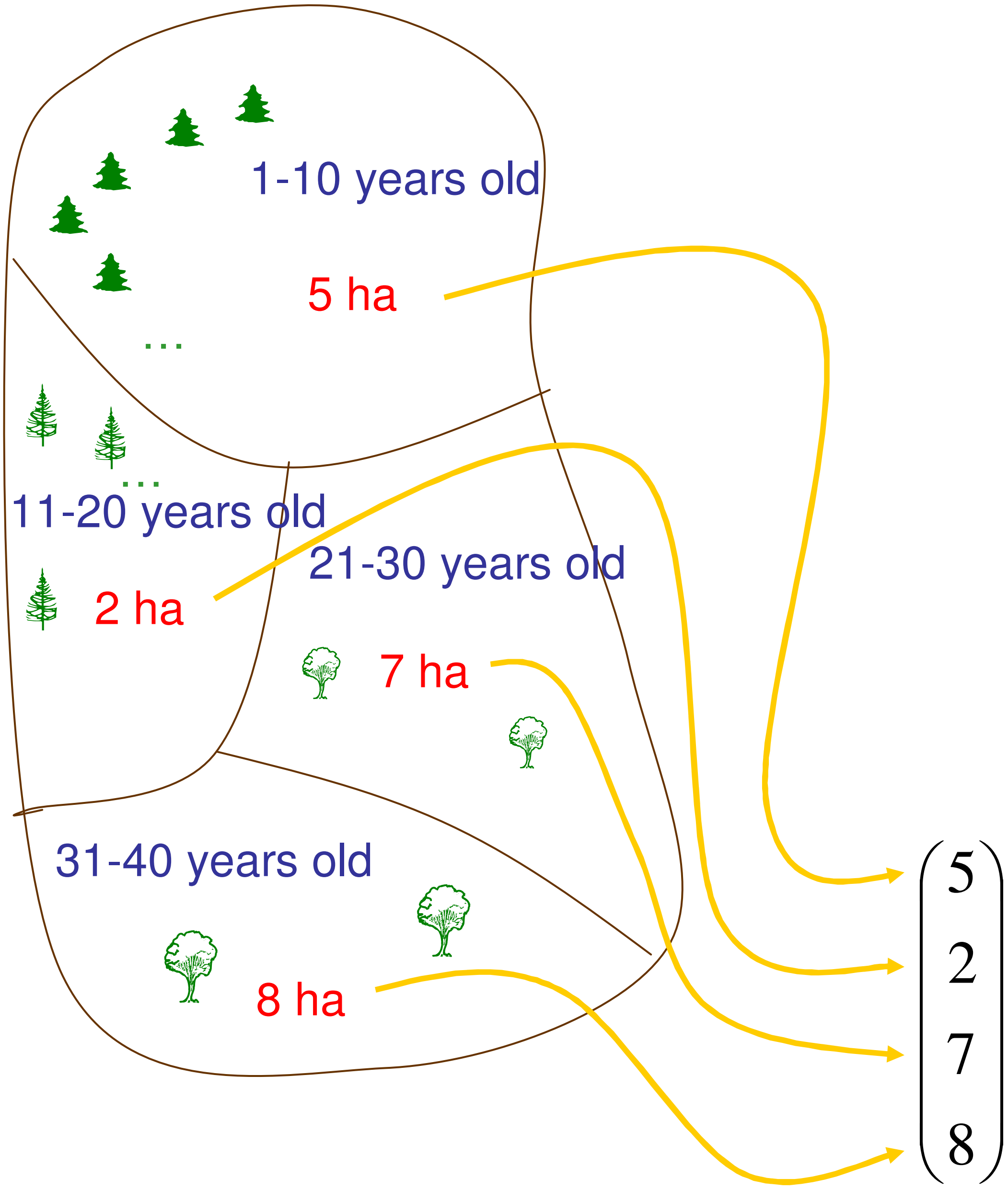


$\mathrm{IR}^3 = \mathrm{IR} \times \mathrm{IR} \times \mathrm{IR}$ 3-dimensional space.

IR^n is called an *n-dimensional vector space*.

Example of a vector in a higher-dimensional vector space IR^n ($n > 3$) :

The age-class vector of a population (e.g., of a forest stand)



Equality of vectors:

Two vectors are equal iff all their corresponding components are equal.

$$\begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{pmatrix} \quad \Leftrightarrow \quad a_1 = b_1 \quad \wedge \quad a_2 = b_2 \quad \wedge \cdots \quad \cdots \wedge \, a_n = b_n$$

Addition of vectors:

Definition of the sum of two vectors in IR^n

$$\begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{pmatrix} + \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{pmatrix} = \begin{pmatrix} a_1 + b_1 \\ a_2 + b_2 \\ \vdots \\ a_n + b_n \end{pmatrix}$$

Properties of the addition of vectors:

$\vec{a} + \vec{b} = \vec{b} + \vec{a}$ commutativity

$(\vec{a} + \vec{b}) + \vec{c} = \vec{a} + (\vec{b} + \vec{c})$ associativity

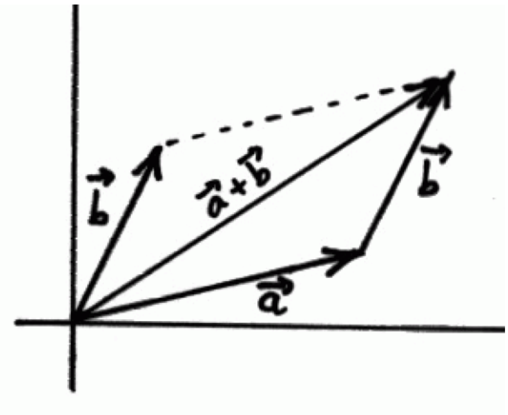
$\vec{a} + \vec{0} = \vec{a},$ neutral element $\vec{0}$

where $\vec{0}$ is the zero vector:

$$\vec{0} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \in \mathrm{IR}^n$$

Geometrical interpretation of vector addition:

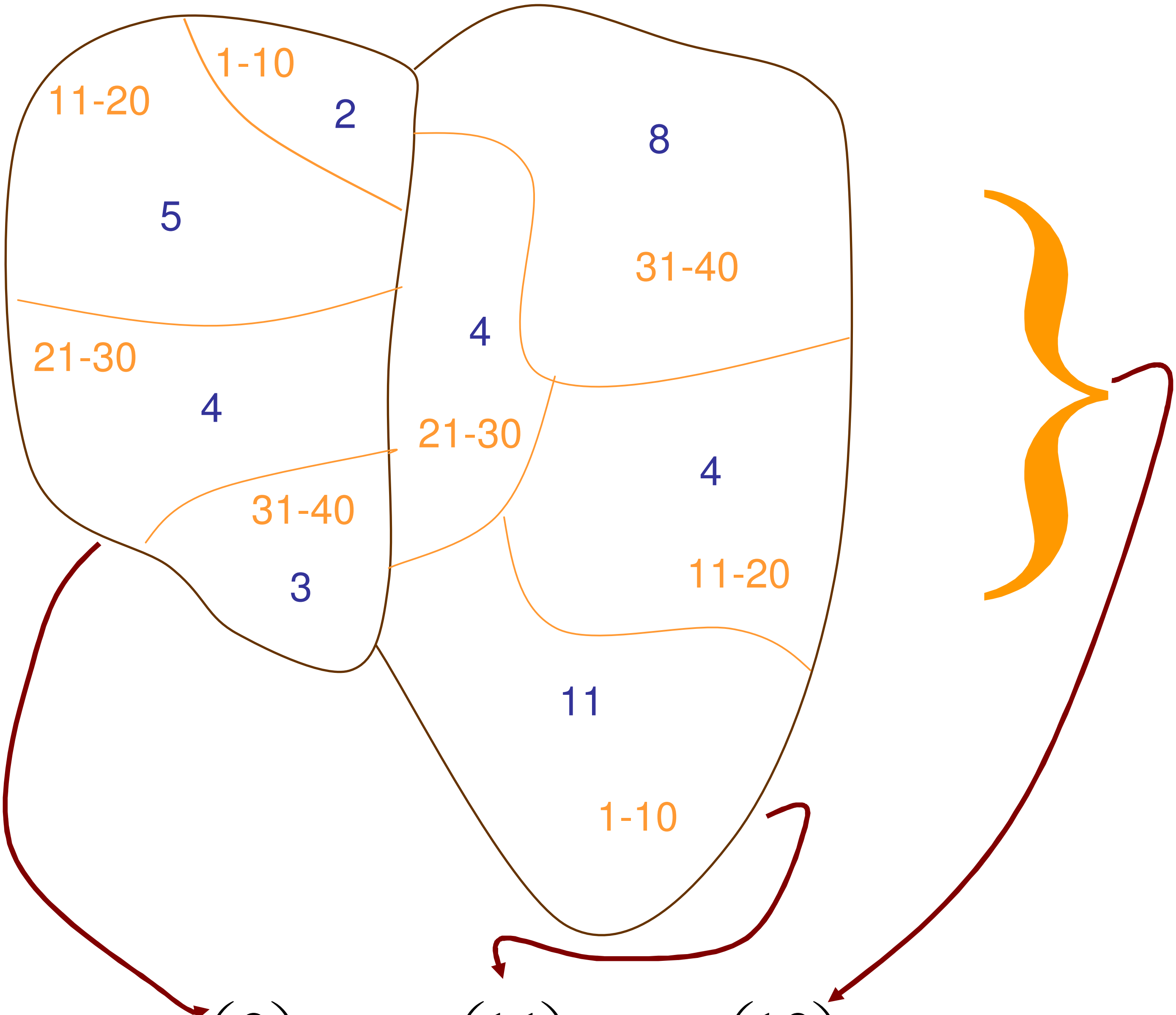
The arrows of both vectors are placed one after the other, and the origin is connected with the new end point.



(in physics: "parallelogram of forces")

The sum in the case of age-class vectors:

aggregation of two forest stands into one.



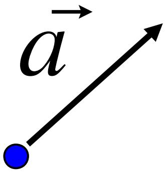
$$\begin{pmatrix} 2 \\ 5 \\ 4 \\ 3 \end{pmatrix} + \begin{pmatrix} 11 \\ 4 \\ 4 \\ 8 \end{pmatrix} = \begin{pmatrix} 13 \\ 9 \\ 8 \\ 11 \end{pmatrix}$$

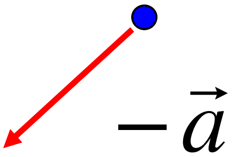
age-class structure of the total area

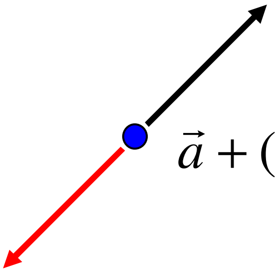
For all vectors $\vec{a}$ from IR^n, there exists exactly one vector $-\vec{a}$ which fulfills $\vec{a}+(-\vec{a})=\vec{0}$.

↑
inverse (negative) element

$$-\vec{a} \;=\; ?$$







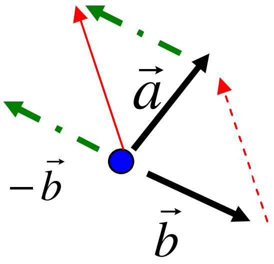
$$\vec{a}+(-\vec{a})=\vec{0}$$

$$=\begin{pmatrix}0\\0\\0\end{pmatrix}$$

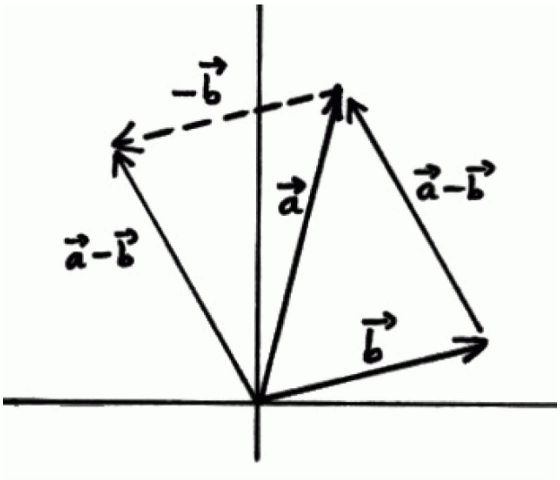
Difference of vectors:

$\vec{a}-\vec{b}$

$=\vec{a}+(-\vec{b})$ (as in the case of real numbers)



Geometrical interpretation of the difference of vectors:



$$\vec{a}-\vec{b}=\vec{a}+(\underset{\uparrow}{-}\vec{b})$$

inversion of the direction

we get thus the "connecting vector" of the endpoints of both vectors.

Multiplication of a vector with a scalar
(≠ „inner product“, ≠ „vector product“ !)

$m \in \mathrm{IR}$, $\vec{a} \in \mathrm{IR}^n$

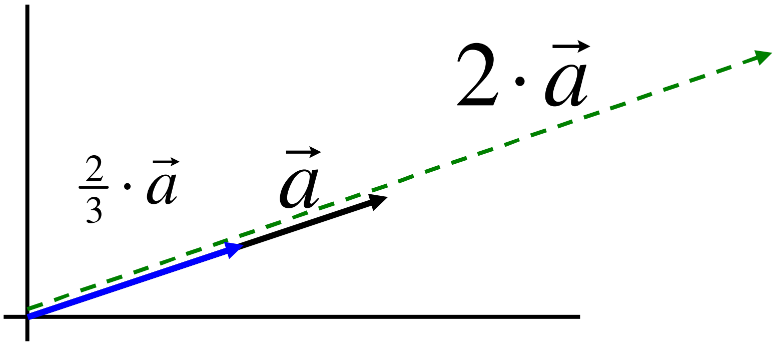
$$m \cdot \vec{a} = m \cdot \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{pmatrix} := \begin{pmatrix} m \cdot a_1 \\ m \cdot a_2 \\ \vdots \\ m \cdot a_n \end{pmatrix} \in \mathrm{IR}^n$$

Example:

$$\tfrac{2}{3} \cdot \begin{pmatrix} 9 \\ -5 \\ 3 \end{pmatrix} = \begin{pmatrix} \tfrac{2}{3} \cdot 9 \\ \tfrac{2}{3} \cdot (-5) \\ \tfrac{2}{3} \cdot 3 \end{pmatrix} = \begin{pmatrix} 6 \\ -\tfrac{10}{3} \\ 2 \end{pmatrix}$$

geometrical meaning:

expansion, resp. compression of $\vec{a}$ by the factor *m*



The direction is inverted, if the factor *m* is < 0.

We have the following rules:

$$1 \cdot \vec{a} = \vec{a}$$

$$0 \cdot \vec{a} = \vec{0}$$

$$(-1) \cdot \vec{a} = -\vec{a}$$

$$m \cdot \vec{0} = \vec{0}$$

$$m \cdot \vec{a} = \vec{0} \quad \Rightarrow \quad m = 0 \vee \vec{a} = \vec{0}$$

$$\left.\begin{aligned} m \cdot (\vec{a} + \vec{b}) &= m \cdot \vec{a} + m \cdot \vec{b} \\ (k + m) \cdot \vec{a} &= k \cdot \vec{a} + m \cdot \vec{a} \end{aligned}\right\} \text{distributive laws}$$

In the following, terms of the form

$$m_1 \cdot \vec{a}_1 + m_2 \cdot \vec{a}_2 + \ldots + m_k \cdot \vec{a}_k$$

$$(= \sum_{i=1}^{k} m_i \cdot \vec{a}_i), \quad m_i \in \mathbb{R}, \quad \vec{a}_i \in \mathbb{R}^n$$

are important. We speak of a linear combination of the vectors $\vec{a}_i, \ldots, \vec{a}_k$; the m_i are called coefficients.

Example (in 3-dimensional space):

$$\vec{a}_1 = (1, -1, 0) \quad , \quad \vec{a}_2 = (2, 1, 1) \quad , \quad \vec{a}_3 = (-2, 0, 0) \quad ,$$

$$\vec{a}_4 = (0, -2, 2)$$

(here written as row vectors for convenience)

The vector

$$\vec{b}=3\,\vec{a}_1-2\,\vec{a}_2+0\,\vec{a}_3+3\,\vec{a}_4$$

is a linear combination of these four vectors.
In column-vector notation, we calculate:

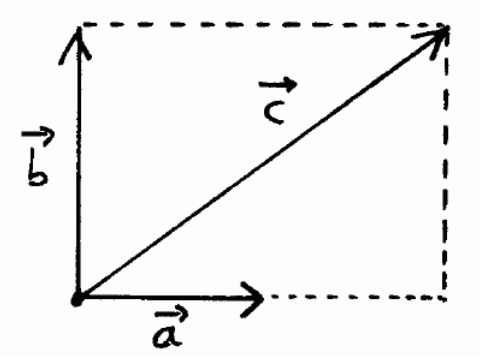
$$\vec{b} \;=\; 3\cdot\begin{bmatrix}1\\-1\\0\end{bmatrix}-2\cdot\begin{bmatrix}2\\1\\1\end{bmatrix}+0\cdot\begin{bmatrix}-2\\0\\0\end{bmatrix}+3\cdot\begin{bmatrix}0\\-2\\2\end{bmatrix} \;=\; \begin{bmatrix}-1\\-11\\4\end{bmatrix}$$

The trivial linear combination

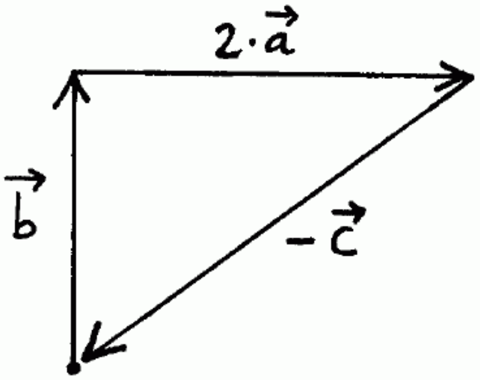
A linear combination is called *trivial* if all coefficients m_1, ..., m_k are 0.
It is called nontrivial if at least one coefficient is *not* 0.

A trivial linear combination has the zero vector as its result.
Can the zero vector also be the result of a *nontrivial* linear combination?

An example: 3 vectors in a plane



We can indeed construct a "cycle" of multiples of these vectors which gives as its sum the zero vector:



$$1\cdot\vec{b}+2\cdot\vec{a}+(-1)\cdot\vec{c}=\vec{0}$$

This is a *nontrivial* linear combination giving the zero vector!
$0\cdot\vec{b}+0\cdot\vec{a}+0\cdot\vec{c}=\vec{0}$ would be trivial.

We say: $\vec{a},\vec{b},\vec{c}$ are *linearly dependent*.

Definition:
Linear dependence / independence of vectors

Given are $k \in \mathbb{N}$ and the vectors
$\vec{a}_1, \vec{a}_2, \ldots, \vec{a}_k \in \mathbb{R}^n$.
These vectors are called *linearly dependent*, if there exist real numbers m_1, ..., m_k , which are *not all equal to zero*, such that

$$\sum_{i=1}^{k} m_i \vec{a}_i = \vec{0} .$$

If the latter equation holds only if all coefficients are 0, then the vectors are called *linearly independent*.

One can prove: Several vectors are linearly dependent if and only if *one of them can be represented as a linear combination of the others*.

Special cases:

IR^1: only sets with one element, { *a* }, with $a \neq 0$ are linearly independent.

IR^2: $\{\vec{a}_1, \vec{a}_2\}$ is linearly dependent $\Leftrightarrow$ both vectors are on a line through the origin.

IR^3: $\{\vec{a}_1, \vec{a}_2, \vec{a}_3\}$ is linearly dependent $\Leftrightarrow$ all three vectors are in a plane going through the origin of the coordinate system.

How to test a set of vectors for linear dependence

Example: Given are the three vectors (1; 2; 3), (0; –1; 0) and (–1; 2; –2). Are they linearly dependent?

Approach: We have to assume $\sum_{i=1}^{3} m_i \vec{a}_i = \vec{0}$.

Written with column vectors, this means:

$$\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} = m_1 \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} + m_2 \begin{bmatrix} 0 \\ -1 \\ 0 \end{bmatrix} + m_3 \begin{bmatrix} -1 \\ 2 \\ -2 \end{bmatrix}$$

For each component, we obtain an equation, giving together the following system of 3 linear equations:

$$
\begin{array}{lllll}
0= & m_1 & & -m_3 & \Rightarrow \quad m_3=m_1 \\
0= & 2m_1 & -m_2 & +2m_3 & \\
0= & 3m_1 & & -2m_3 & \Rightarrow \quad 2m_3=3m_1
\end{array}
$$

We can solve this step by step for the unknowns m_i. In this case, we obtain quickly $m_1 = m_2 = m_3 = 0$. So the system can only be fulfilled if all coefficients are zero, and the 3 vectors have been proven als *linearly independent*.

Examples for training:

Linearly dependent or independent? Decide yourself!

$$(a)\ \left\{ \begin{bmatrix} 3 \\ 3 \end{bmatrix},\ \begin{bmatrix} -2 \\ -2 \end{bmatrix} \right\}$$

$$(b)\ \left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix},\ \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\}$$

$$(c)\ \left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix},\ \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right\}$$

$$(d)\ \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix},\ \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix},\ \begin{bmatrix} 5 \\ 4 \\ 0 \end{bmatrix} \right\}$$

$$(e)\ \left\{ \begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \end{bmatrix},\ \begin{bmatrix} 1 \\ 0 \\ 2 \\ 3 \end{bmatrix},\ \begin{bmatrix} 1 \\ 1 \\ 3 \\ 3 \end{bmatrix} \right\}$$

Rank of a set of vectors

The number of elements of the *maximal* linearly independent subset of a given set of vectors is called the *rank* of the set of vectors.

The basis of a vector space

IR^n has infinitely many elements.
Is there a finite subset $\{\vec{a}_1, \ldots, \vec{a}_k\}$,
such that all vectors from IR^n
can be represented uniquely as a linear combination of the $\vec{a}_i$?

YES!
Such a set of vectors is called a *basis* of IR^n.

Most simple example of a basis:

$$\vec{e}_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \quad \vec{e}_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \quad \ldots, \vec{e}_n = \begin{pmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 1 \end{pmatrix},$$

the *standard basis* of IR^n.

There are infinitely many bases, which have, however, all the same number of elements (namely, *n*). This number is called the *dimension* of the vector space.

Example:

$$\left\{\begin{pmatrix}1\\1\\0\end{pmatrix},\begin{pmatrix}0\\0\\1\end{pmatrix},\begin{pmatrix}0\\0\\2\end{pmatrix}\right\}$$ has rank 2

lin. indep. *lin. dependent*

If we remove $\begin{pmatrix}0\\0\\2\end{pmatrix}$, we obtain a linearly independent vector system:

$$\left\{\begin{pmatrix}1\\1\\0\end{pmatrix},\begin{pmatrix}0\\0\\1\end{pmatrix}\right\}$$, rank 2.

If we add now, e.g., $\begin{pmatrix}1\\0\\0\end{pmatrix}$, we obtain a basis of IR^3, i.e., a maximal linearly independent subset:

$$\left\{\begin{pmatrix}1\\1\\0\end{pmatrix},\begin{pmatrix}0\\0\\1\end{pmatrix},\begin{pmatrix}1\\0\\0\end{pmatrix}\right\}$$, rank 3.

lin. independent

3 is the dimension of IR^3.

If we add an arbitrary further element,

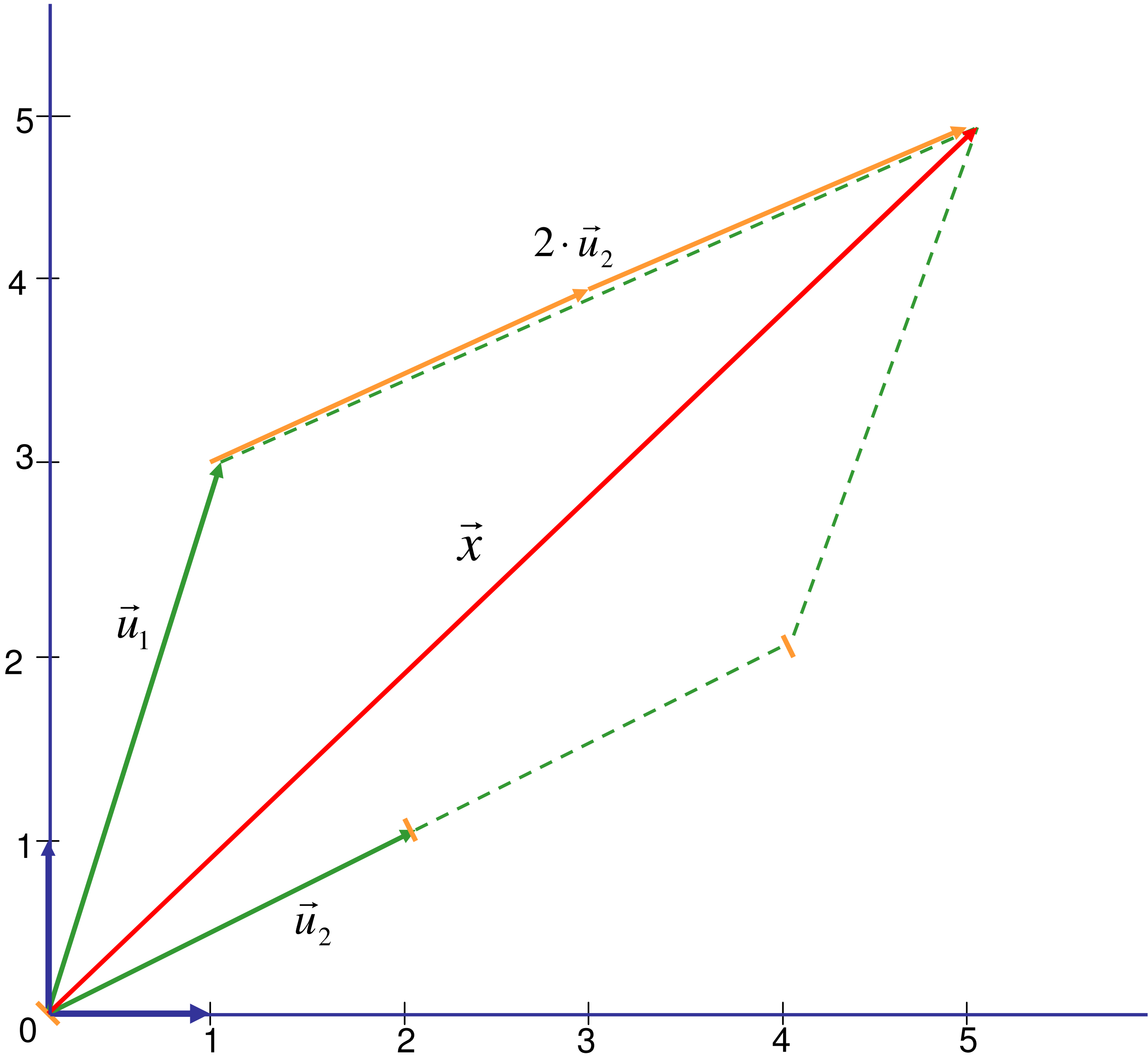
e.g., $\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$, the set becomes linearly dependent:

$$1 \cdot \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} + 0 \cdot \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} + (-1) \cdot \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + (-1) \cdot \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = \vec{0} \; .$$

The *coordinates* of a vector with respect to a given basis

When an arbitrary basis is given, every vector can be expressed uniquely as a linear combination of the elements of this basis (i.e., the coefficients are uniquely determined).

Example:



$$\vec{u}_1 = \begin{pmatrix} 1 \\ 3 \end{pmatrix}, \quad \vec{u}_2 = \begin{pmatrix} 2 \\ 1 \end{pmatrix}, \quad \{\vec{u}_1, \vec{u}_2\}\, basis \;\; of \;\; \mathbb{R}^2$$

$$\vec{x} = \begin{pmatrix} 5 \\ 5 \end{pmatrix} = 1 \cdot \vec{u}_1 + 2 \cdot \vec{u}_2$$

(1; 2) are the coordinates of $\vec{x}$

w.r.t. $\{\vec{u}_1, \vec{u}_2\}$.

In the special case of the standard basis, we have always:

$$a_1 \cdot \vec{e}_1 + a_2 \cdot \vec{e}_2 + \ldots + a_n \cdot \vec{e}_n$$

$$= a_1 \cdot \begin{pmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix} + a_2 \cdot \begin{pmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} + \ldots + a_n \cdot \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix}$$

$$= \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{pmatrix}$$

The <u>components</u> $a_1, \ldots, a_n$ of a vector $\vec{a} \in \mathrm{IR}^n$ are exactly the <u>coordinates</u> of $\vec{a}$ with respect to the standard basis.

The *inner product* of vectors and the *norm* of a vector

The <u>inner product</u> of two vectors

↓

a product of vectors which gives as result a scalar!

Let there be given:

$$\vec{x} = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}, \vec{y} = \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{pmatrix} \in \mathbb{R}^n .$$

We define:

$$\vec{x} \cdot \vec{y} \quad := \quad x_1 \cdot y_1 + x_2 \cdot y_2 + \ldots + x_n \cdot y_n$$

$$= \sum_{i=1}^{n} x_i \cdot y_i \in \mathbb{R}$$

„inner product of $\vec{x}$ and $\vec{y}$“

$\vec{x} \cdot \vec{y}$ is not a vector, thus, e.g., $(\vec{a} \cdot \vec{b}) + \vec{c}$ is <u>senseless</u>.

Example:

$$\begin{pmatrix} 2 \\ 1 \\ 5 \end{pmatrix} \bullet \begin{pmatrix} -1 \\ 3 \\ 8 \end{pmatrix} = 2 \cdot (-1) + 1 \cdot 3 + 5 \cdot 8$$

$$= -2 + 3 + 40$$

$$= 41$$

Significance:

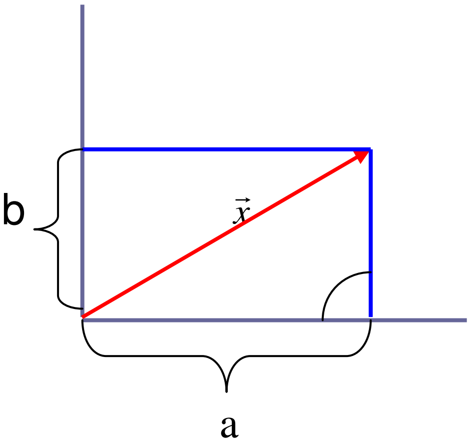
The inner product enables propositions about lengths and angles of vectors.

The (Euclidean) *norm* of $\vec{x} \in$ IR^2 is defined as

$$\left\| \begin{pmatrix} a \\ b \end{pmatrix} \right\| = \sqrt{a^2+b^2}$$

= length of $\vec{x}$ according to Pythagoras.

analogously in IR^3.



geometrical interpretation is thus:

norm = length of the vector (arrow).

The vector $\frac{\vec{x}}{\|\vec{x}\|}$ $(i.e. \frac{1}{\|\vec{x}\|} \cdot \vec{x})$ has length 1.

It is called normed.

General definition of the norm (or length) of a vector:

$$\|\vec{x}\| := \sqrt{\vec{x} \cdot \vec{x}} = \sqrt{\sum_{i=1}^{n} x_i^2}$$

Two vectors $\vec{x}, \vec{y}$ are mutually orthogonal (perpendicular) to each other iff $\vec{x} \cdot \vec{y} = 0$.

Example:

$$\underset{\substack{\uparrow \\ \text{in } xy \text{ plane}}}{\begin{pmatrix} 2 \\ 3 \\ 0 \end{pmatrix}} \cdot \underset{\substack{\nwarrow \\ \text{on } z \text{ axis}}}{\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}} = 2 \cdot 0 + 3 \cdot 0 + 0 \cdot 1 = 0$$

Generally, in IR^n the *angle formula* holds:

$$\sphericalangle(\vec{x}, \vec{y}) = \arccos \frac{\vec{x} \cdot \vec{y}}{\| \vec{x} \| \cdot \| \vec{y} \|}$$

The cross product of vectors in IR^3

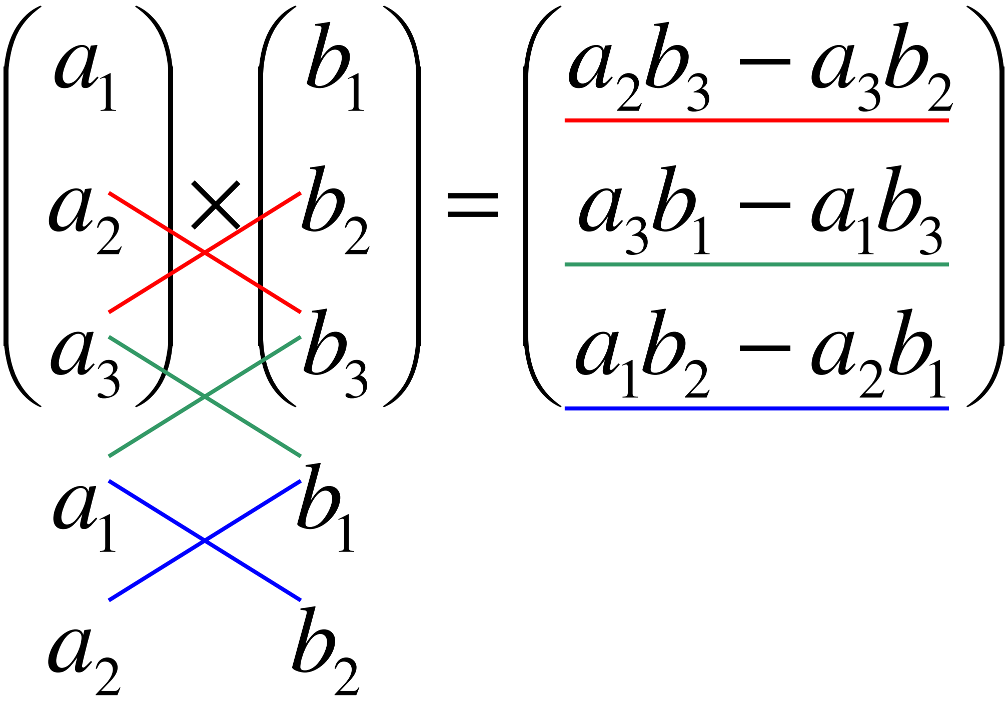
Let there be given two 3-dimensional vectors

$$\vec{a}=\begin{bmatrix}a_1\\a_2\\a_3\end{bmatrix}, \quad \vec{b}=\begin{bmatrix}b_1\\b_2\\b_3\end{bmatrix} \in \mathbb{R}^3$$

The *vector product* or *cross product* $\vec{a}\times\vec{b}$ of both vectors is defined as the following new 3-dimensional vector:

$$\vec{a}\times\vec{b}:=\begin{bmatrix}a_2b_3-a_3b_2\\a_3b_1-a_1b_3\\a_1b_2-a_2b_1\end{bmatrix} \in \mathbb{R}^3 \quad .$$

Rule for memorizing the components of the cross product:



$$\begin{pmatrix}a_1\\a_2\\a_3\end{pmatrix}\times\begin{pmatrix}b_1\\b_2\\b_3\end{pmatrix}=\begin{pmatrix}a_2b_3-a_3b_2\\a_3b_1-a_1b_3\\a_1b_2-a_2b_1\end{pmatrix}$$

The cross product has the following properties:

$\vec{b}\times\vec{a}=-\vec{a}\times\vec{b}$ (thus, in general, the factors must not be flipped)

$\vec{a}\times\vec{b}=\vec{0} \quad \Leftrightarrow \quad \{\vec{a},\vec{b}\}$ *linearly dependent*

$\vec{a}\times\vec{b}$ stands always *orthogonal* to $\vec{a}$ and $\vec{b}$ (so this is an easy way to find some vector orthogonal to a plane if it is needed)

$\vec{a} \;,\; \vec{b} \;,\; \vec{a}\times\vec{b}$ form in this order a "right-hand system" (orientated like the first three fingers of the right hand)

$\|\vec{a}\times\vec{b}\|=\|\vec{a}\|\cdot\|\vec{b}\|\cdot\sin\sphericalangle(\vec{a},\vec{b})$

= *area of the parallelogram* which is spanned by $\vec{a}$ and $\vec{b}$

Attention:
The cross product does *only* exist in IR^3 !