Computer Science and Mathematics

Part I:
Fundamental Mathematical Concepts

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1. Mathematical Logic

Propositions

- can be either true or false
- Examples: "Vienna is the capital of Austria", "Mary is pregnant", "3+4=8"
- can be combined by logical operators, e.g., "Today is Tuesday *and* the sun is shining".

Usual logical operators and their abbreviations:

$a \wedge b$	a and b	(**A**nd)
$a \vee b$	a or b	(latin: **v**el)
$\neg a$	not a	
$a \Rightarrow b$	a implies b (if a then b)	
$a \Leftrightarrow b$	a is equivalent to b (if and only if a then b; iff a then b)	

Quantifiers

$\forall x$ for all x holds ...
$\exists x$ there exists an x for which ...

Further symbols

$:=$ is equal by definition
$:\Leftrightarrow$ is equivalent by definition

2. Sets

A *set* is a collection of different objects, which are called the *elements* of the set.
The order in which the elements are listed does not matter.
A set can have a finite or an infinite number of elements. We speak of finite and infinite sets.

Examples:
The set of all human beings on earth (finite)
The set of all prime numbers (infinite)

Sets are usually designated by upper-case letters, their elements by lower-case letters.

$a \in M$ *a* is element of the set *M*
$a \notin M$ *a* is *not* element of the set *M*

Two notations for sets:

- Listing of all elements, delimited by commas (or semicolons) and put in braces:
  $A = \{ 1; 2; 3; 4; 5 \}$

- Usage of a variable symbol and specification of a proposition (containing the variable) which has to be fulfilled by the elements:
  $A = \{ x \mid x \text{ is a positive integer smaller than } 6 \}$
  (the vertical line is read: "... for which holds: ...")

  alternative notation for the last one:
  $\mathrm{A} = \{ x \in \mathbb{N} \mid x < 6 \}$

($\mathbb{N}$ is the set of positive integer numbers, not including 0.)

Number of elements of a set M (also called *cardinality* of M): $|\mathrm{M}|$

example: $|\{ x \in \mathbb{N} \mid x < 12 \wedge x \text{ is even} \}| = 5$

Propositions involving sets:
Example: $\exists n \in \mathbb{N} : n^2 = 2^n$
(true, because it is fulfilled for $n = 2$)

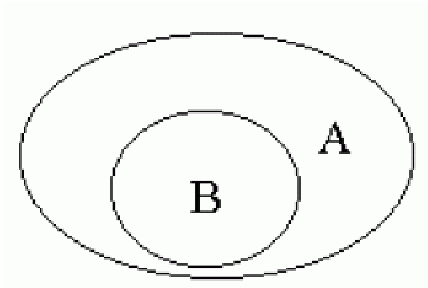
A special set: The *empty set*
Notation: $\varnothing$
For the empty set, we have $|\varnothing| = 0$.

Subsets and supersets

If A contains all elements of B (and possibly some more), B is called a *subset* of A (and A a *superset* of B).

Notation: $B \subseteq A$ (or, equivalently, $A \supseteq B$)

Visualization by a so-called *Venn diagram*:



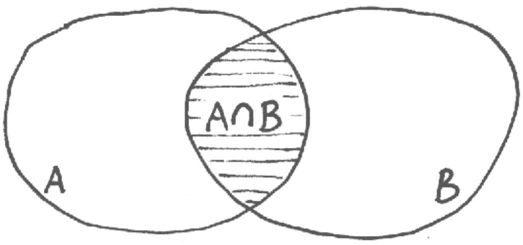
It holds: $A \subseteq B \;\wedge\; B \subseteq A \;\Leftrightarrow\; A = B$.

Intersection

The <u>intersection</u> of the sets A and B is the set of all elements which are element of A <u>and</u> of B.

Operator symbol: $\cap$

$$A \cap B = \{ x \mid x \in A \wedge x \in B \}$$



example:

$\{1;2;3;4\} \cap \{2;4;6;8\}$

$= \{2;4\}$

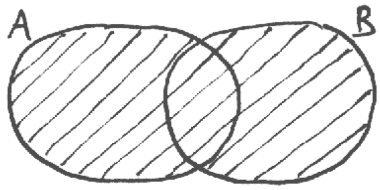
Two sets A and B are called <u>disjoint</u> if $A \cap B = \emptyset$.

Union

The <u>union</u> of the sets A and B is the set of all elements which are element of A <u>or</u> of B.

Operator symbol: $\cup$ ($\cup$ = Union)

$$A \cup B = \{ x \mid x \in A \vee x \in B \}$$



example:

$\{1;2;3;4\} \cup \{2;4;6;8\}$

$= \{1;2;3;4;6;8\}$

What is the number of elements $|A \cup B|$?

If A and B are disjoint, we have:

$$|A \cup B| = |A| + |B|$$

Generalization:

If $A_1, A_2, \ldots, A_n$ are all pairwise disjoint ($\forall j,k: A_j \cap A_k = \emptyset$ if $j \neq k$), then $|A_1 \cup A_2 \cup \ldots \cup A_n| = |A_1| + |A_2| + \ldots + |A_n|$.

Remarks:

1. $(A \cup B) \cup C = A \cup (B \cup C)$ (associativity), so we can omit the parentheses (the same holds for + and for $\cap$)

(2) Short notations for iterated operations:

for n sets $A_1, \ldots, A_n$: $\displaystyle\bigcup_{i=1}^{n} A_i := A_1 \cup A_2 \cup \ldots \cup A_n$

for n numbers $x_1, \ldots, x_n$: $\displaystyle\sum_{i=1}^{n} x_i := x_1 + x_2 + \ldots + x_n$

$$\prod_{i=1}^{n} x_i := x_1 \cdot x_2 \cdot \ldots \cdot x_n$$

(3) The formula $|A \cup B| = |A| + |B|$ does not hold if A and B are not disjoint. In the general case, we have:

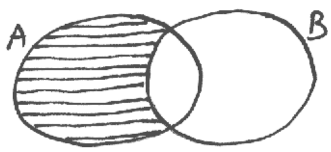
$$|A \cup B| = |A| + |B| - |A \cap B|.$$

Difference of sets

The <u>difference set</u> of the sets A and B is the set of all elements which are element of A <u>but not</u> of B. ("A without B")

Operator symbol: $-$ (sometimes also used: $\setminus$)

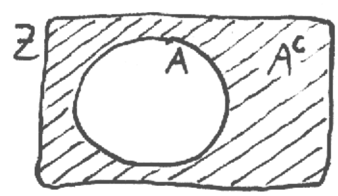
$$A - B = \{ x \mid x \in A \wedge x \notin B \}$$



example:
$\{1; 2; 3; 4\} - \{2; 4; 6; 8\}$
$= \{1; 3\}$

Complement

If all considered sets are subsets of a given basic set Z, the difference $Z - A$ is often called the <u>complement</u> of A and is denoted A^c.



$$A^c = \{ x \in Z \mid x \notin A \}$$

The power set

The set of all subsets of a given set S is called the *power set* of S and is denoted $\mathrm{P}(S)$.

$$P(S) = \{ A \mid A \subseteq S \}$$

Example:

$$S = \{1; 2; 3\}$$

$$P(S) = \{ \emptyset; \{1\}; \{2\}; \{3\}; \{1;2\}; \{1;3\}; \{2;3\}; \{1;2;3\} \}$$

For the number of its elements, we have always:

$$|P(S)| = 2^{|S|}$$

Cartesian product of sets

The <u>cartesian product</u> of two sets A and B, denoted $A \times B$, is the set of all possible <u>ordered pairs</u> where the first component is an element of A and the second component an element of B.

$$A \times B = \{ (a, b) \mid a \in A \wedge b \in B \}$$

Remark: In an ordered pair, the order of the components is fixed. If $a \neq b$, then $(a, b) \neq (b, a)$.

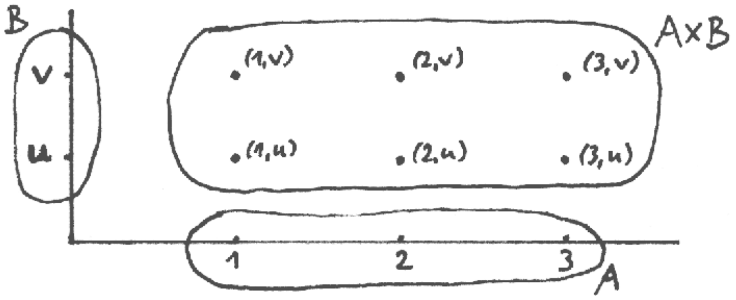
Example: $A = \{ 1; 2; 3 \}$, $B = \{ u; v \}$:

$$A \times B = \{ (1,u); (2,u); (3,u); (1,v); (2,v); (3,v) \}$$

Attention: Usually it is $A \times B \neq B \times A$!

Number of elements: $|A \times B| = |A| \cdot |B|$

Visualization of $A \times B$ in a coordinate system:



If A and B are subsets of the set $\mathbb{R}$ of real numbers, we can use the well-known cartesian coordinate system.

Products of more than two sets

The elements of $(A\times B)\times C$ are "nested pairs" $((a,b),c)$; we identify them with the triples $\underline{(a,b,c)}$ and write $A\times B\times C$. Analogously for quadruples, etc.

$$A_1\times A_2\times\ldots\times A_n = \{(a_1,a_2,\ldots,a_n) \mid a_1\in A_1 \wedge a_2\in A_2 \wedge\ldots\wedge a_n\in A_n\}$$

If $A_1 = A_2 = \ldots = A_n$, we write:

$$A^n = \underbrace{A\times A\times\ldots\times A}_{n \text{ times}}$$

= set of all <u>n-tuples</u> with components from A.

Example:

$B = \{x; y\} \quad\Rightarrow$

$$B^3 = \{(x,x,x); (x,x,y); (x,y,x); (x,y,y); (y,x,x); (y,x,y); (y,y,x); (y,y,y)\}$$

If the components are letters, the parentheses and commas are often omitted: $B^3 = \{xxx; xxy; xyx; \ldots; yyy\}$ set of <u>words</u> of length 3

Set of arbitrary words (strings) over a set:

$$A^* = A^0 \cup A^1 \cup A^2 \cup A^3 \cup \ldots$$

with $A^0 := \{\varepsilon\}$, where ε is the <u>empty word</u>.

Example: $\{x; y\}^* = \{\varepsilon; x; y; xx; xy; yx; yy; xxx; \ldots\}$

$A^+ = A^1 \cup A^2 \cup A^3 \cup \ldots$ does not contain the empty word.

The cartesian product in the description of datasets

Frequently, informations regarding a measurement are put together in an n-tuple. Example:

S = set of time values
T = set of temperature values
U = set of laboratory identifiers
V = set of measurement values

A measurement is then represented by a 4-tuple

$$(s, t, u, v) \in S\times T\times U\times V$$

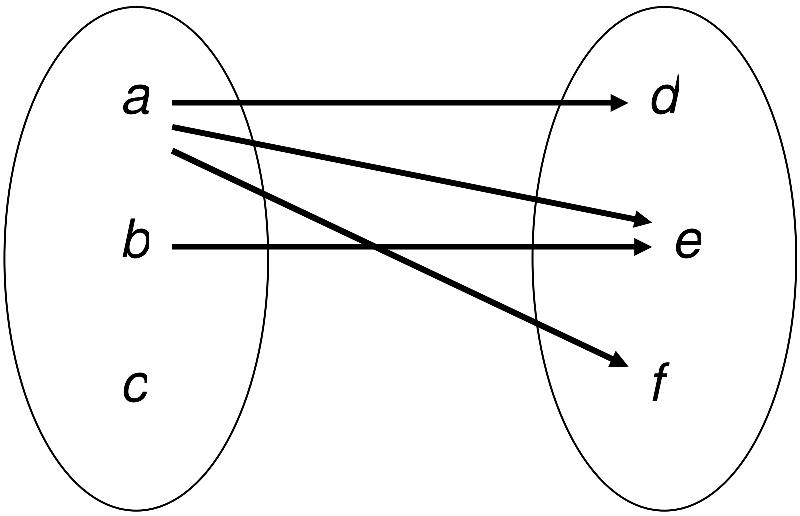
time of measurement — current temperature — lab id — measured value

3. Relations

A (binary) relation R between two sets A and B is a subset of $A \times B$.
That means, a relation is represented by a set R of ordered pairs (a, b) with $a \in A$ and $b \in B$.
If $(a, b) \in R$ we write also $a\ R\ b$ (infix notation).

Graphical representation (if A and B are finite):

If $(a, b) \in R$, connect a and b by an arrow



The converse relation R^{-1} of R:

$(b, a) \in R^{-1} \Leftrightarrow (a, b) \in R$

R^{-1} is a subset of $B \times A$.

In the graphical representation, switch the directions of all arrows to obtain the converse relation!

If $A = B$, we have a relation *in* a set A.

Example: $A = \mathbb{R}$ (set of real numbers), $R = <$ relation "smaller as". R consists of all number pairs (x, y) with $x < y$.

Generalization: n-ary relation:
any subset of $A_1 \times A_2 \times ... \times A_n$.

4. Graphs

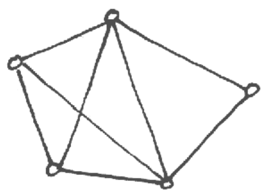
A *graph* consists of a set V of *vertices* and a set E of *edges*. Each edge connects two vertices.

Different variants of graphs differ in the way how the edges are defined and what edges are allowed:

- Undirected graphs:

  The edges are (unordered) 2-element subsets of V.

  Visualization by undirected arcs:

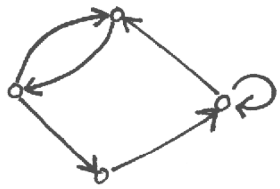


- Directed graphs:

  The edges are ordered pairs, i.e., $E \subseteq V \times V$

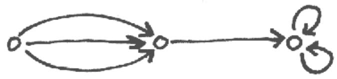
  (E is a relation in V)

  Visualization by directed arcs. "Loops" are allowed, multiple arcs between the same vertices are not allowed:



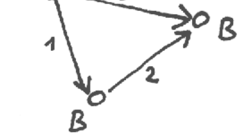
- Multigraphs:

  Multiple directed edges are allowed



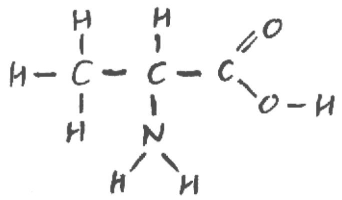
- Labelled graphs:

  Vertices and/or edges have labels from a set of vertex/edge labels (names, numbers, ...)



Examples :

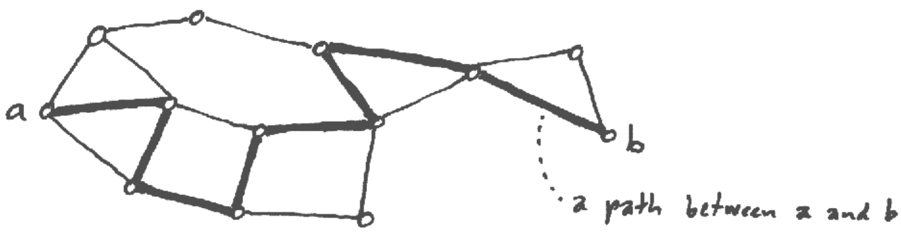
- transport networks
- metabolic networks
- food webs
- class diagrams in software engineering
- genealogical trees
- structural formulae in chemistry



vertex-labelled multigraph

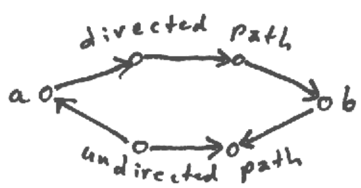
Paths in graphs

A <u>path</u> is a sequence of edges where two consecutive edges have one vertex in common :

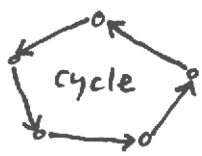


A path where start- and end vertex coincide is called a <u>circle</u>.

In directed graphs, we distinguish between directed and undirected paths.

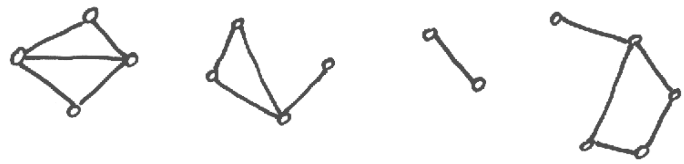


A directed circle is called a <u>cycle</u>.



Connectedness

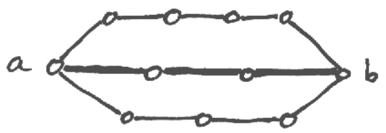
If for every pair of vertices (a, b) in a graph, there is a path between a and b, the graph is called <u>connected</u>.
Every unconnected graph can be decomposed in <u>connected components</u>.



a graph with 4 connected components

Graph-theoretical distance

The <u>distance</u> between two vertices a and b in a graph is the length, i.e., the number of edges, of the shortest path between a and b — if such a path exists. Otherwise, the distance is undefined.

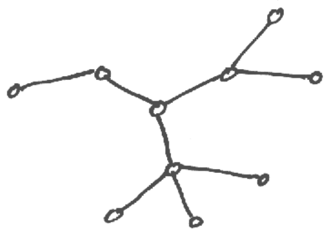


dist(a, b) = 3

Trees

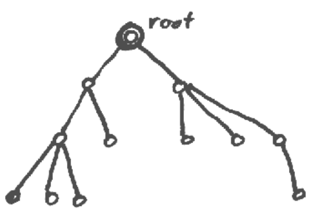
A <u>tree</u> is a graph without circles.

A tree:

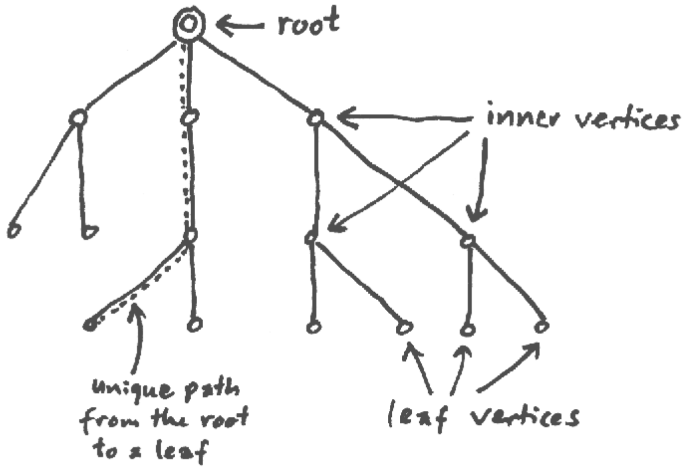


Example: phylogenetic trees,
describing genetic kinship between species

A <u>rooted tree</u> is a tree in which one vertex, the root, is distinguished.
The root is often drawn at the top:



Rooted trees are used to describe hierarchies, e.g., in biological systematics, in organizations or in nested directories of data.



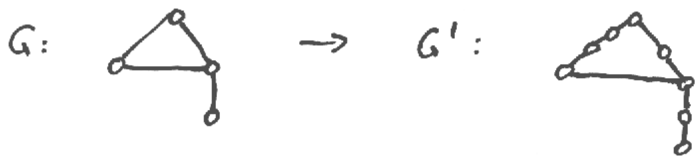
Degree

The number of edges to which a vertex belongs is called the <u>degree</u> of the vertex.

In directed graphs we distinguish between <u>indegree</u> and <u>outdegree</u> of a vertex.

Subdivision

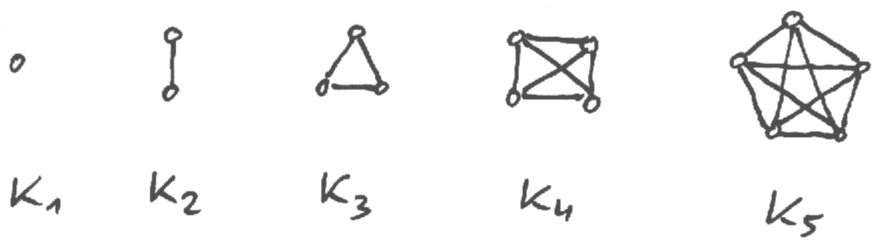
A <u>subdivision</u> G' of a graph G is obtained by inserting vertices of degree 2 in the edges of G.



Complete graphs

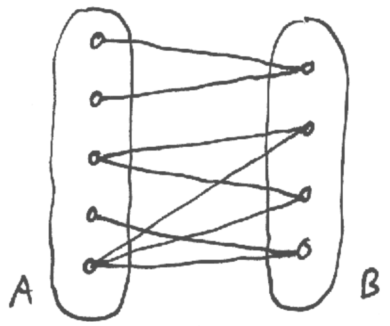
The <u>complete graph</u> K_n is the graph with n vertices where every pair of different vertices is connected by an edge.

(Also called: <u>Clique</u>.)



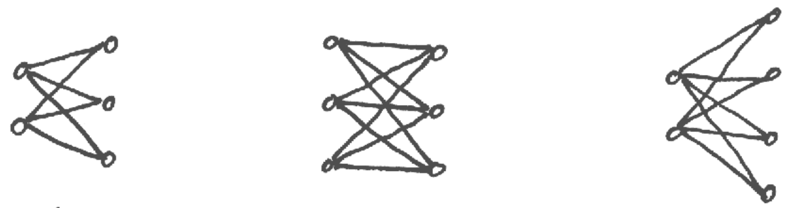
Bipartite graphs

A <u>bipartite graph</u> can be split into two disjoint sets of vertices, A and B, such that all edges go from a vertex from A to a vertex from B.



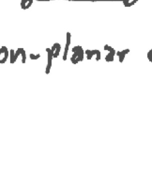
(The edges then form a relation between A and B.)

The <u>complete bipartite graph</u> $K_{m,n}$ is a bipartite graph with $|A| = m$, $|B| = n$, and edges go from every vertex of A to every vertex of B.

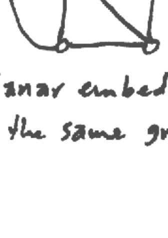


Planarity

A graph is <u>planar</u> if its vertices and edges can be embedded in the plane, with edges as arcs in the plane, such that no two different edges intersect in points different from their start- and end vertex.



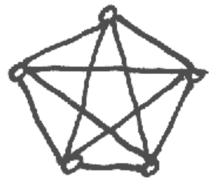
non-planar embedding



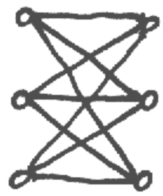
planar embedding of the same graph

Kuratowski's theorem:

A graph is planar if and only if it does not contain any subdivision of K_5 or $K_{3,3}$.



K_5



$K_{3,3}$

5. Functions

The *function* is a fundamental notion in mathematics. It is used to describe:

- a dependency between two variables (e.g., between measured sizes of the same objects)
- a transformation of data during some calculation or processing step

$$\text{input} \longrightarrow f \longrightarrow \text{output}$$

- a development of a variable in time or in space (e.g., heigth growth of a plant; magnetic field strength in space...)

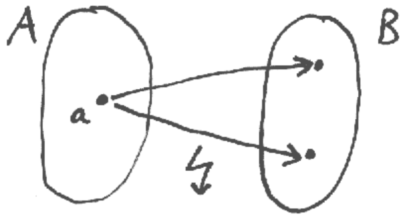
Frequently used synonyms for *function*:
mapping, *transformation*, *operator*

The precise definition of a function identifies it with the relation between "input" (argument(s)) and "output" (value), i.e., a function is defined as a special case of a relation:

A relation R between the sets A (= possible input values) and B (= possible values) is a *function* if for every $a \in A$ there is exactly one $b \in B$ with $a\ R\ b$.
We write then f instead of R and use frequently the notation $f(a) = b$.
Further typical notations:

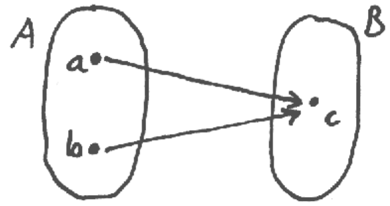
$$f: A \to B, \quad a \mapsto b\,.$$

The following situation is thus excluded for functions, because a would have two different "images" in B:



$f(a)$ must be <u>unique</u>.

Allowed is:



$f(a) = c$

$f(b) = c$

written as set: $f = \{(a,c); (b,c)\} \subseteq A \times B$

We say: "f maps a to c", "c is an image of a under f".

f is the function, $f(a)$ is a special value.

a is called the <u>argument</u> of f.

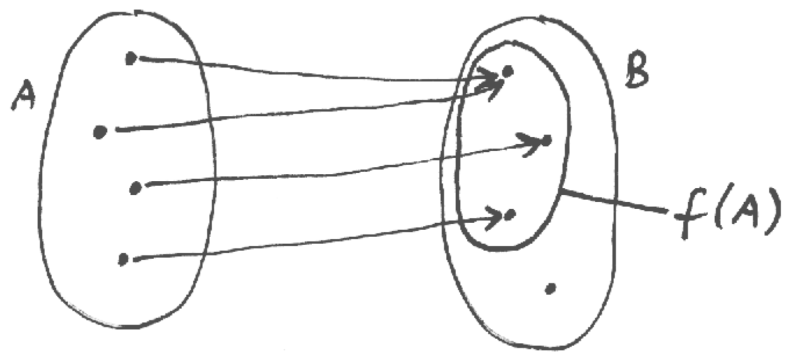
Different notations:

$f(a)$ or fa prefix notation

af postfix notation

Domain and image of a function

$f: A \to B$



A is called the <u>domain</u> of f

$f(A)$ is called the <u>image</u> of A under f,

sometimes also <u>range</u> of f

Multivariate functions

Functions can have several arguments:

$$f: A \times B \to C$$

$$(a, b) \mapsto f(a, b) = c \in C$$

$a \in A \quad b \in B$

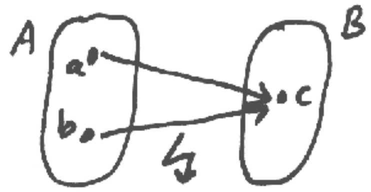
$$f: A_1 \times \ldots \times A_n \ni (a_1, \ldots, a_n) \mapsto f(a_1, \ldots, a_n) \in C$$

Injective, surjective, bijective functions

Injectivity

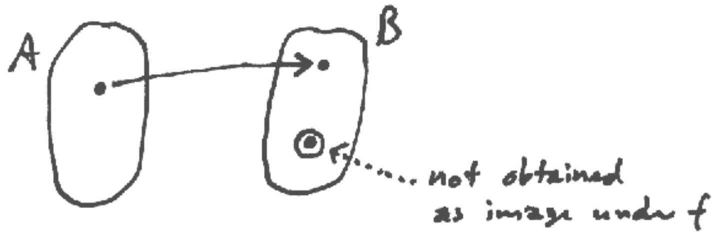
A function $f: A \to B$ is called <u>injective</u> if
$\forall a, b \in A: \quad a \neq b \Rightarrow f(a) \neq f(b).$

That means, two distinct elements of A have always distinct images. Not allowed is:



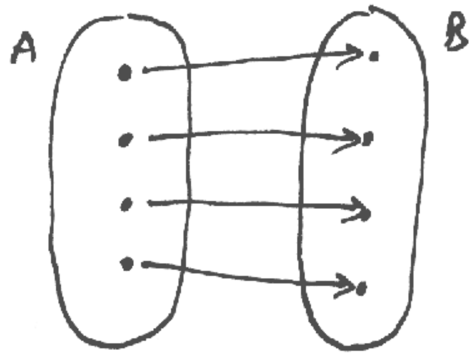
Surjectivity

A function $f: A \to B$ is called <u>surjective</u> if
$\forall b \in B \; \exists a \in A: f(a) = b.$ All elements of B are images of elements of A. Not allowed is:



Bijectivity

$f: A \to B$ is called <u>bijective</u> if it is injective and surjective.



Bijective functions can be <u>inverted</u>,

i.e., the converse relation $f^{-1}: B \to A$ is again a function.

That means: $f^{-1}(b)$ is <u>unique</u> for every $b \in B$.

Example where this is not the case:

$f(x) = x^2 \qquad A = B = \mathbb{R}$

$$\left.\begin{array}{l} f(2) = 4 \\ f(-2) = 4 \end{array}\right\} \Rightarrow f^{-1}(4) \text{ not unique, } f^{-1} \text{ no function}$$

f is not bijective on $\mathbb{R}$.

How to obtain the inverse function of a bijective real-valued function (with one argument):

- solve $f(x) = y$ for x, so you obtain $x = f^{-1}(y)$
- switch the names of the variables ($x \leftrightarrow y$)