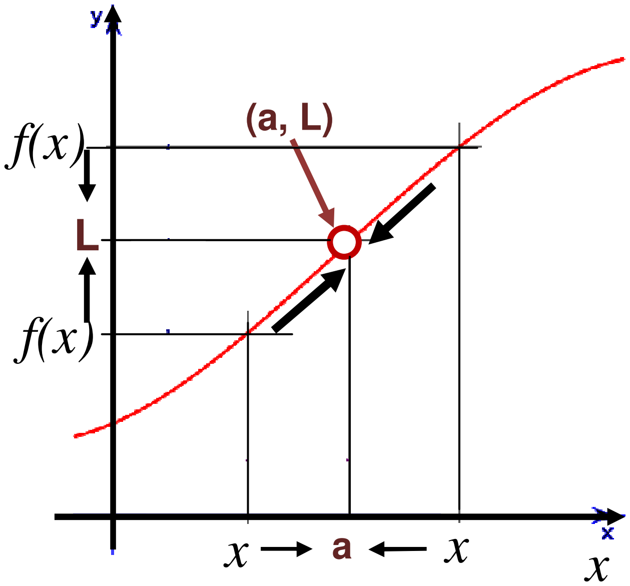
Limits of Functions



Informal Definition:

If the values of $f(x)$ can be made as close to L as we like by taking values of x sufficiently close to a [but not equal to a] then we write

$$\lim_{x \to a} f(x) = L$$

or

$$f(x) \to L \text{ as } x \to a$$

Observe:

- "$x \to a$" means x can approach a from either side
- On a sketch, the graph of $f(x)$ approaches the 2-D plane location [destination] called (a, L), but the graph itself may have no point $(a, f(a))$ occupying that location!

L may not be $f(a)$

The language to describe how the outputs $f(x)$ behave as the inputs x approaches a number L

$$\lim_{x \to a} f(x) = L$$

Example:

$$\lim_{x \to 0} \left[\frac{x}{\sqrt{x+1} - 1} \right] -?$$

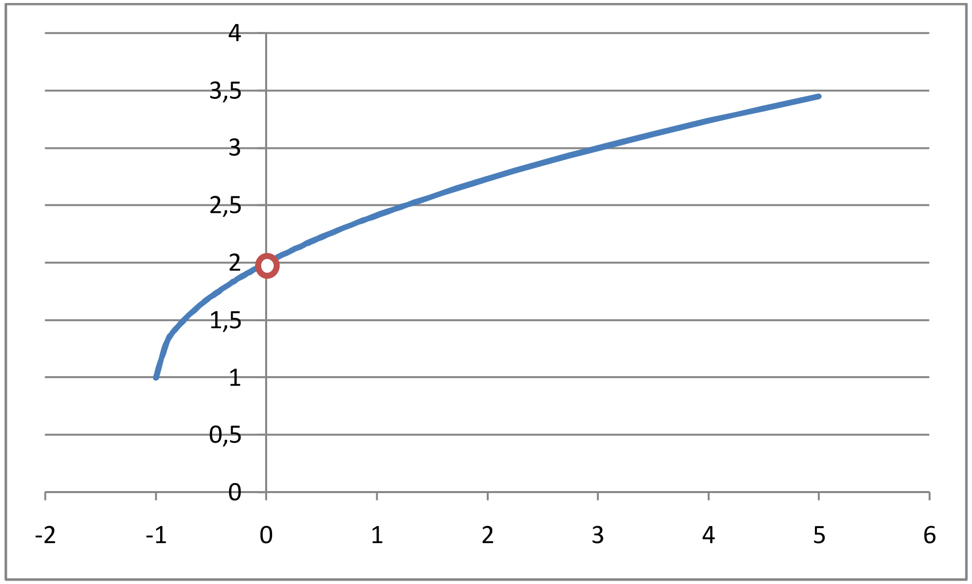
We examine the graph

Domain of $f(x)$

$$\sqrt{x+1} - 1 \neq 0, x \neq 0$$

$$x + 1 \geq 0, x \geq -1$$

$$\{x \in \mathbb{R} | x \geq -1, x \neq 0\}$$



Conjecture:

$$\lim_{x\to 0}\left[\frac{x}{\sqrt{x+1}-1}\right]=2$$

General definition:

Let $f(x)$ be a function and a a real number (that may be or may be not in the domain of f). We say that the limit as x approaches a of $f(x)$ is L, written

$$\lim_{x \to a} f(x) = L$$

if $f(x)$ can be made arbitrarily close to L by choosing x sufficiently close to (but not equal to) a.

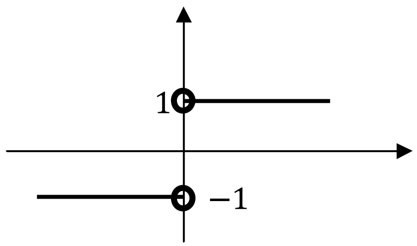
If no such number exists, then we say that

$\lim_{x \to a} f(x)$ does not exist .

Warning: Not all limits exist!

Example:

$$f(x) = \frac{|x|}{x} = \begin{cases} 1 \; if \; x > 0 \\ -1 \; if \; x < 0 \end{cases}$$



- $x \to 0$ from the left, $f(x) \to -1$
- $x \to 0$ from the right, $f(x) \to 1$

So $\lim_{x\to 0} f(x)$ has no meaning!

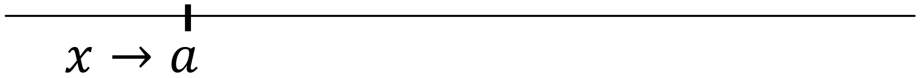
Two-Sided and One-Sided Limits

Notation

"x approaches a from the left"

$x \to a^-$ [minus in a superscript position] or $x \uparrow a$ [comes up to a] or $x \nearrow a$

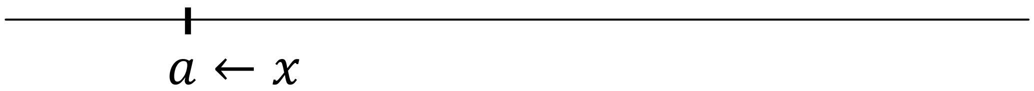
$$\lim_{x \to a^-} f(x) = L$$



"x approaches a from the right"

$x \to a^+$ [plus in a superscript position] or $x \downarrow a$ [comes down to a] or $x \searrow a$

$$\lim_{x \to a^+} f(x) = L$$



Relationship between Two-Sided and One-Side Limits: Theorem

$$\lim_{\underbrace{x \to a}_{\text{two-sided}}} f(x) = L$$

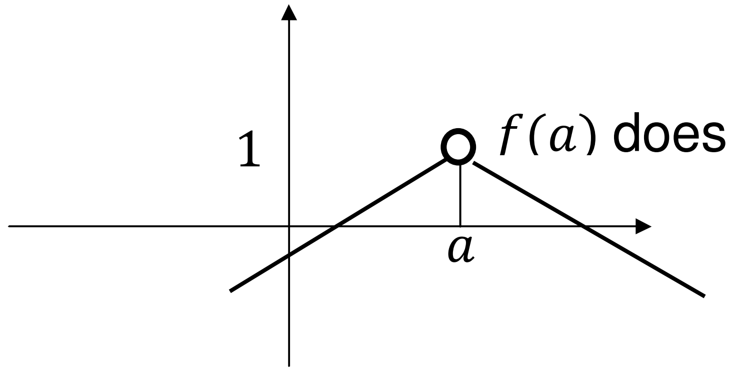
$\Leftrightarrow$ [if and only if]:

$\lim_{x \to a^-} f(x) = L$ exists

$\lim_{x \to a^+} f(x) = L$ exists

and both equal L

Example:



$$\lim_{x\to a^-} f(x) = 1 = \lim_{x\to a^+} f(x)$$

The Algebra of Limits as $x \to a$

Basic Limits as $x \to a, a^+, a^-$ used in polynomial functions and rational functions.

<u>The constant function</u>

$$\lim_{x \to a}(k) = k$$

<u>The identity function: $f(x)$</u>

$$\lim_{x \to a}(x) = a$$

<u>The reciprocal ("flip over") function:</u> $f(x) = \frac{1}{x}$

$$\lim_{x \to 0^-}\left(\frac{1}{x}\right) = -\infty$$

$$\lim_{x \to 0^+}\left(\frac{1}{x}\right) = \infty$$

Limits of Sums, Differences, Products, Quotients and Roots

The "Rules" of Algebra for Limits

Let a be any real number and

$$\lim_{x\to a} f(x) = L_1$$

$$\lim_{x\to a} g(x) = L_2$$

then

$$\lim_{x\to a}[f(x) + g(x)] = \lim_{x\to a} f(x) + \lim_{x\to a} g(x) = L_1 + L_2$$

$$\lim_{x\to a}[f(x) - g(x)] = \lim_{x\to a} f(x) - \lim_{x\to a} g(x) = L_1 - L_2$$

$$\lim_{x\to a}[f(x) \cdot g(x)] = \lim_{x\to a} f(x) \cdot \lim_{x\to a} g(x) = L_1 \cdot L_2$$

$$\lim_{x\to a}|f(x)| = \left|\lim_{x\to a} f(x)\right| = |L_1|$$

$$\lim_{x\to a}[kf(x)] = k \cdot \lim_{x\to a} f(x) = k \cdot L_1$$

$$\lim_{x\to a}[f(x)^n] = \left[\lim_{x\to a} f(x)\right]^n = \mathrm{L}_1^n$$

$$\lim_{x\to a}\left[\frac{f(x)}{g(x)}\right] = \frac{\lim_{x\to a} f(x)}{\lim_{x\to a} g(x)} = \frac{L_1}{L_2}$$

Provided $L_2 \neq 0$

$$\lim_{x\to a} \sqrt[n]{f(x)} = \sqrt[n]{\lim_{x\to a} f(x)} = \sqrt[n]{L_1}$$

Provided when n= even then $L_1 \geq 0$

Limits of Polynomial Function

Polynomial Expressions

A monomial (one-term polynomial) has the form

$$a_n \cdot x^n$$

n=0,1,2, 3,…
not negative
called the **degree** of the monomial

A real number constant called a “coefficient”
Subscript n is a label

x- a variable

Two monomial with the same degree and same variable are called “like terms”:
$a_n \cdot x^n$; $b_n \cdot x^n$ - “like terms”

A polynomial in one variable has the standard form: [higher powers → lower powers]

$$a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0$$

$a_n \neq 0$ leading coefficient

By the “Rules of Algebra” for Limits we can break down **polynomials** into simpler parts

Example:

$$\lim_{x \to a}[f(x) \pm g(x)] = \lim_{x \to a} f(x) \pm \lim_{x \to a} g(x)$$

$$\lim_{x \to a}[f(x)^n] = \left[\lim_{x \to a} f(x)\right]^n$$

$$\lim_{x \to a}(k) = k$$

$$\lim_{x \to a}(x) = a$$

$$\lim_{x \to 5}[x^2 - 4x + 3] = \lim_{x \to 5}[x^2] - \lim_{x \to 5}[4x] + \lim_{x \to 5}[3] = \left(\lim_{x \to a}[x]\right)^2 - 4\lim_{x \to a}[x] + 3$$

$$\lim_{x \to a}[kf(x)] = k \cdot \lim_{x \to a} f(x)$$

$$= 5^2 - 4 \cdot 5 + 3 = 8$$

For any polynomial function

$$\lim_{x \to a} p(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0 = p(a)$$

For polynomial, this limit is the same as “ substitution of a for x

Limits of Rational Functions $\frac{p(x)}{q(x)}$ and the appearance of $\frac{0}{0}$

There are 3 cases to consider

Case 1: $q(a) \neq 0$ Limit $= \frac{p(a)}{q(a)}$

Example:

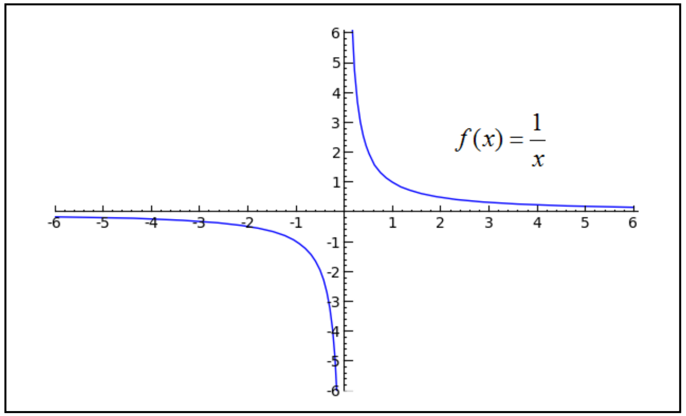
$$\lim_{x\to 2}\left[\frac{5x^3+4}{x-3}\right]\begin{matrix}\leftarrow p(x)\\ \leftarrow q(x)\end{matrix} \quad a = 2 =$$

$$= \frac{\lim_{x\to 2}[5x^3+4]}{\lim_{x\to 2}[x-3]} = \frac{\overbrace{5\cdot 2^3+4}^{p(a)}}{\underbrace{2-3}_{q(a)\neq 0}} = -44$$

Case 2: $p(a) \neq 0 \; and \; q(a) = 0$ Limit does not exist (division by 0!)

Classic Examples:

$$f(x) = \frac{\overbrace{1}^{p(x)}}{\underbrace{x-a}_{q(x)}}$$



$$\lim_{x \to a^-} \left[\frac{1}{x-a}\right] = -\infty$$

$$\lim_{x \to a^+} \left[\frac{1}{x-a}\right] = +\infty$$

a)

$$\lim_{x \to a} \left[\frac{1}{(x-a)^2} \right] = \infty$$

b)

$$\lim_{x \to a} \left[\frac{-1}{(x-a)^2} \right] = -\infty$$

Case 3: $p(a) = 0 \; and \; q(a) = 0$ Limit $\frac{p(a)}{q(a)} = \frac{0}{0}$ an indeterminate form: We cannot determine whether the limit exists or not, without more work!

Example:

$$\lim_{x \to 2} \left[\frac{x^2 - 4}{x - 2} \begin{matrix} \leftarrow p(x) \\ \leftarrow q(x) \end{matrix} \right] \; both \; p(2) = 0 = q(2)$$

$$\boxed{Factor + cancel}$$

$$\lim_{x \to 2} \left[\frac{(x-2)(x+2)}{x-2} \right] = \lim_{x \to 2} [(x+2)] = 4$$

This is only one particularly technique! Does not work always!

The Algebra of Limits as $x \to \pm\infty$: End Behavior

Basic Limits:

The constant function

$$\lim_{x \to -\infty} (k) = k$$

and

$$\lim_{x \to \infty} (k) = k$$

The identity function: $f(x)$

$$\lim_{x \to -\infty} (x) = -\infty$$

$$\lim_{x \to \infty} (x) = \infty$$

The reciprocal ("flip over") function: $f(x) = \frac{1}{x}$

$$\lim_{x \to -\infty} \left(\frac{1}{x}\right) = 0$$

$$\lim_{x \to \infty} \left(\frac{1}{x}\right) = 0$$

Limits of Sums, Differences, Products, Quotient and Roots

The "Rules" of Algebra for Limits applied to $x \to -\infty$ or $x \to \infty$

We only state for $x \to \infty$ case

As before, suppose:

$$\lim_{x\to\infty} f(x) = L_1$$

$$\lim_{x\to\infty} f(x) = L_2$$

then

$$\lim_{x\to\infty}[f(x) + g(x)] = \lim_{x\to\infty} f(x) + \lim_{x\to\infty} g(x) = L_1 + L_2$$

$$\lim_{x\to\infty}[f(x) - g(x)] = \lim_{x\to\infty} f(x) - \lim_{x\to\infty} g(x) = L_1 - L_2$$

$$\lim_{x\to\infty}[f(x) \cdot g(x)] = \lim_{x\to\infty} f(x) \cdot \lim_{x\to\infty} g(x) = L_1 \cdot L_2$$

$$\lim_{x\to a}|f(x)| = \left|\lim_{x\to a} f(x)\right| = |L_1|$$

$$\lim_{x\to\infty}[kf(x)] = k \cdot \lim_{x\to\infty} f(x) = k \cdot L_1$$

$$\lim_{x\to\infty}[f(x)^n] = \left[\lim_{x\to\infty} f(x)\right]^n = \mathrm{L}_1^n$$

$$\lim_{x\to\infty}\left[\left(\frac{1}{x}\right)^n\right] = \left[\lim_{x\to\infty}\frac{1}{x}\right]^n = 0$$

$$\lim_{x\to\infty}\left[\frac{f(x)}{g(x)}\right] = \frac{\lim_{x\to\infty} f(x)}{\lim_{x\to\infty} g(x)} = \frac{L_1}{L_2}$$

Provided $L_2 \neq 0$

$$\lim_{x\to\infty}\sqrt[n]{f(x)} = \sqrt[n]{\lim_{x\to\infty} f(x)} = \sqrt[n]{L_1}$$

Provided when n= even then $L_1 \geq 0$

Limits of Polynomial Functions: Two End Behaviors

A polynomial function

$$f(x) = c_n x^n + c_{n-1} x^{n-1} + \cdots + c_1 x + c_0$$

Where $c_n \neq 0$

The "two end behaviors" are that as $x \to \infty$ (the rightward end) or $x \to -\infty$ (the leftward end)

Then

$$\left.\begin{matrix} f(x) \to \infty \\ f(x) \to -\infty \end{matrix}\right\} \text{ The two possibilities}$$

Observe:

$$f(x) = c_n x^n + c_{n-1} x^{n-1} + c_1 x + c_0 = x^n \left[c_n + \underbrace{\frac{c_{n-1}}{x} + \cdots + \frac{c_1}{x^{n-1}} + \frac{c_0}{x^n}}_{go\ to\ 0!} \right]$$

So, the "end behavior" of $f(x)$ matches the "end behavior" of $c_n x^n$

Theorem:

$$\lim_{x \to \pm\infty} [c_n x^n + c_{n-1} x^{n-1} + c_1 x + c_0] = \lim_{x \to \pm\infty} [c_n x^n]$$

Example:

$$\lim_{x \to -\infty} [-4x^8 + 17x^5 + 3x^4 + 2x - 50] = \lim_{x \to -\infty} [-4x^8] = -\infty$$

Limits of Rational Functions: Three Types of End Behavior

$$f(x) = \frac{p(x)}{q(x)} \begin{matrix} \leftarrow top \\ \leftarrow botton \end{matrix}$$

The Degree of a polynomial is the exponent of the highest power of x in the polynomial

Type 1. Deg(top)=Deg(bottom)

$$\lim_{x \to \infty} f(x) = \frac{leading\ coefficient\ of\ top}{leading\ coefficient\ of\ bottom}$$

Example:

$$f(x) = \frac{-x}{7x + 4}$$

$$\lim_{x \to \infty} f(x) = \lim_{x \to \infty} \left[\frac{-x}{7x + 4}\right] = \lim_{x \to \infty} \left[\frac{-1}{7 + \underbrace{\frac{4}{x}}_{\searrow 0}}\right] = -\frac{1}{7} = \frac{l.c.of\ top}{l.c.of\ bottom}$$

Type 2. Deg(top)<Deg(bottom)

$$\lim_{x\to\infty} f(x) = 0$$

Example:

$$f(x) = \frac{5x+2}{2x^3-1}$$

$$\lim_{x\to\infty} f(x) = \lim_{x\to\infty}\left[\frac{5x+2}{2x^3-1}\right] = \lim_{x\to\infty}\left[\frac{\frac{5}{x^2}+\frac{2}{x^3}}{2-\frac{1}{x^3}}\right] = 0$$

Always zero

$y = 0$ the (x-axis) is a horizontal asymptote

Type 3. Deg(top)>Deg(bottom)

If $leading\ coefficient\ of\ top > 0$

$$\left.\begin{matrix} \infty & if & x \to \infty \\ -\infty & if & x \to -\infty \end{matrix}\right\}$$ Always one of these

If $leading\ coefficient\ of\ top < 0$

$$\left.\begin{matrix} \infty & if & x \to -\infty \\ -\infty & if & x \to \infty \end{matrix}\right\}$$ Always one of these

Example:

$$f(x) = \frac{x^2 + 4x + 5}{x - 1}$$

$$\lim_{x\to\infty} f(x) = \lim_{x\to\infty}\left[\frac{x^2 + 4x + 5}{x - 1}\right] = \lim_{x\to\infty}\left[\frac{1 + \frac{4}{x} + \frac{5}{x^2}}{\frac{1}{x} - \frac{1}{x^2}}\right] = \infty$$

Limits of $\ln(x)$ and e^x

$$\lim_{x\to\infty}[lnx] = \infty$$

note, that $\lim_{x\to-\infty} lnx$ makes no sense

$$\lim_{x\to 0^+}[lnx] = -\infty$$

$$\lim_{x\to\infty}[e^x] = \infty$$

$$\lim_{x\to-\infty}[e^x] = 0$$

$$\lim_{x\to\infty}[e^{-x}] = 0$$

$$\lim_{x\to-\infty}[e^{-x}] = -\infty$$

Limits of Trigonometric Functions

$$\lim_{x \to 0} cosx = cos0 = 1$$

$$\lim_{x \to 0} sinx = sin0 = 0$$

$$\lim_{x \to \infty} arctanx = \frac{\pi}{2}$$

$$\lim_{x \to -\infty} arctanx = -\frac{\pi}{2}$$

Some more techniques for computing limits

a) Rational functions: divide the top and the bottom to cancel (reduce): factor and cancel

$$\lim_{x\to 1}\frac{x^3-3x+2}{x^3-x^2-x+1}=\lim_{x\to 1}\frac{(x^2-2x+1)(x+2)}{(x^2-2x+1)(x+1)}=\lim_{x\to 1}\frac{x+2}{x+1}=\frac{1+2}{1+1}=\frac{3}{2}$$

b)If we have some roots: expand the top and the bottom with a factor: The standard thing to do with a square root in a sum or difference is <u>rationalize</u>

$$\lim_{x\to 0}\frac{\sqrt{x-1}-2}{x^2-4x-5}=\lim_{x\to 5}\frac{\left(\sqrt{x-1}-2\right)\left(\sqrt{x-1}+2\right)}{(x^2-4x-5)\left(\sqrt{x-1}+2\right)}$$

$$=\lim_{x\to 1}\frac{x-5}{(x+1)(x-5)\left(\sqrt{x-1}+2\right)}=\lim_{x\to 1}\frac{1}{(x+1)\left(\sqrt{x-1}+2\right)}=\frac{1}{4}$$

c) One-sided limits:

$$\lim_{x\to a^+} f(x),\ or\ \lim_{x\to a^-} f(x),$$

Substitution $x = a + \delta, x = a - \delta$, replace $x \to a^+$ and $x \to a^-$ by $\delta \to 0$

$$\lim_{\delta\to 0^+} f(a+\delta)$$

$$\lim_{\delta\to 0^-} f(a-\delta)$$

$$\lim_{x\to 3^+} \frac{2x+1}{9-x^2} = \|Substitution: x = 3+\delta\ \ x\to 3\ \ \delta\to 0\|$$

$$\lim_{\delta\to 0} \frac{2(3+\delta)+1}{9-(3+\delta)^2} = \lim_{\delta\to 0} \frac{6+2\delta+1}{9-(9+6\delta+\delta^2)} = \lim_{\delta\to 0} -\frac{7+2\delta}{6\delta-\delta^2} = -\infty$$

$$\lim_{x\to 3^-} \frac{2x+1}{9-x^2} = \|Substitution: x = 3-\delta\ \ x\to 3\ \ \delta\to 0\|$$

$$= +\infty$$

Finding a Limit by "Squeezing"

Old problem: How do we calculate

$$\lim_{x \to 0} \frac{sinx}{x} \quad \left[the\ function\ \frac{sinx}{x}\ does\ not\ exist\ at\ x = 0\right]$$

Answer: "we squeeze"

Theorem ["squeezing" Theorem]

If $g(x) \leq f(x) \leq h(x)$ for all $x \neq c$ in some open interval containing c and

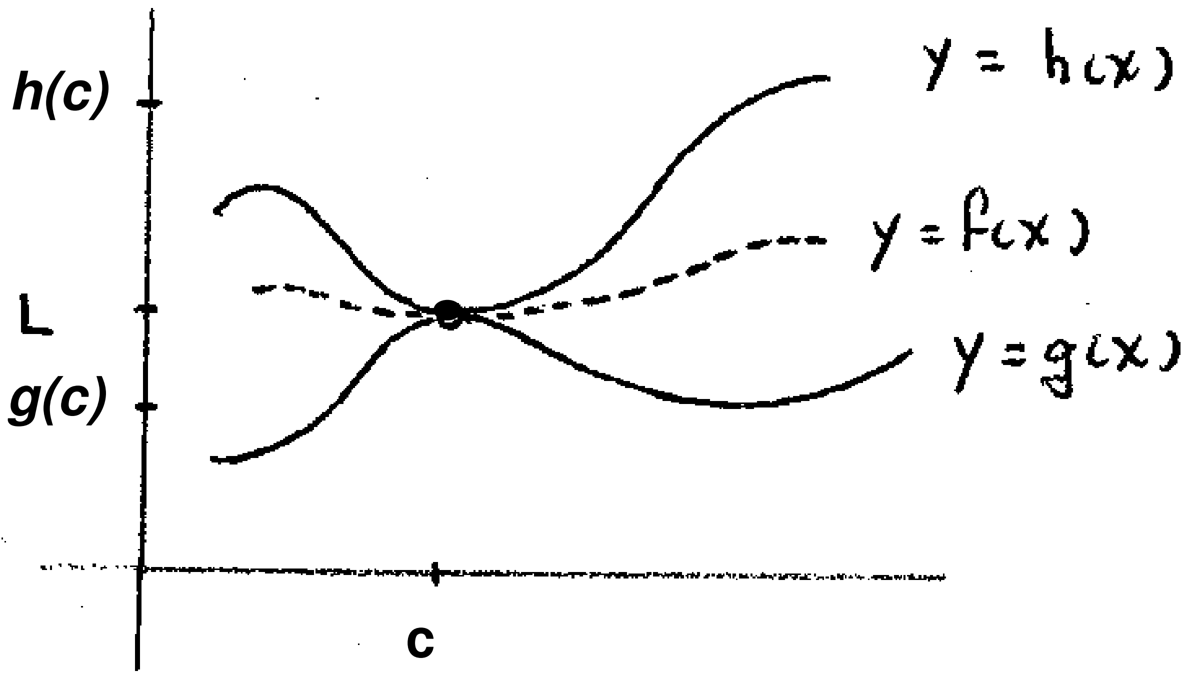
$$\lim_{x \to c} g(x) = L = \lim_{x \to c} h(x)$$

Then

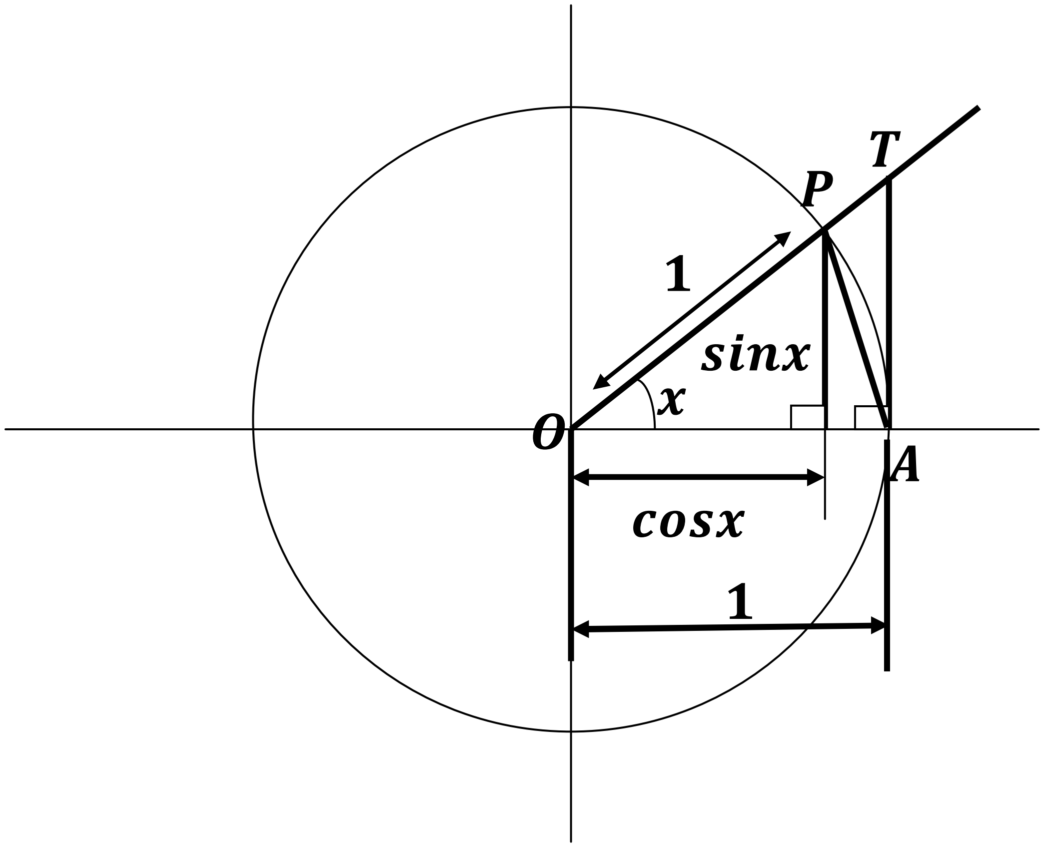
$$\lim_{x \to c} f(x) = L$$

too.

Idea:



$$\lim_{x \to 0} \frac{sinx}{x} = ?$$



$$\frac{sinx}{cosx} = \frac{TA}{1}$$

$$\text{Area } (\Delta = OAP) \le Area\ Sector(OAP) \le Area(\Delta OAT)$$

$$\frac{1}{2} \cdot 1 \cdot sinx \le \frac{x}{2\pi}(\pi \cdot 1^2) \le \frac{1}{2} \cdot 1 \cdot \frac{sinx}{cosx}$$

$$\left[Multiply\ by\ \frac{2}{sinx}\right]$$

$$1 \le \frac{x}{sinx} \le \frac{1}{cosx}$$

$$1 \ge \frac{sinx}{x} \ge cosx$$

Since

$$\lim_{x\to 0} cosx = cos0 = 1 \; and \; \lim_{x\to 0} 1 = 1$$

The squeeze Theorem implies, that

$$\lim_{x\to 0} \frac{sinx}{x} = 1$$

Exercise 1

$$\lim_{x\to 0}\left[\frac{1-cosx}{x}\right]=\lim_{x\to 0}\left[\frac{1-cos}{x}\cdot\frac{1+cosx}{1+cosx}\right]=\lim_{x\to 0}\left[\frac{1-cos^2x}{x(1+cosx)}\right]=$$

$$\lim_{x\to 0}\left[\frac{sin^2x}{x(1+cosx)}\right]=\lim_{x\to 0}\left[\frac{sinx}{x}\cdot\frac{sinx}{1+cosx}\right]=1+\frac{0}{1+1}=0$$

Exercise 2

$$\lim_{x\to 0}\left[\frac{tan(7x)}{sin(3x)}\right] = \lim_{x\to 0}\left[\frac{sin(7x)}{cos(7x)\cdot sin(3x)}\right]$$

$$= \frac{7}{3}\lim_{x\to 0}\left[\frac{sin(7x)}{7x}\cdot\frac{1}{cos(7x)}\cdot\frac{3x}{sin(3x)}\right] = \frac{7}{3}\cdot 1\cdot 1\cdot 1 = \frac{7}{3}$$

Continuous Function

Continuous Function at a single point $\boldsymbol{x = c}$

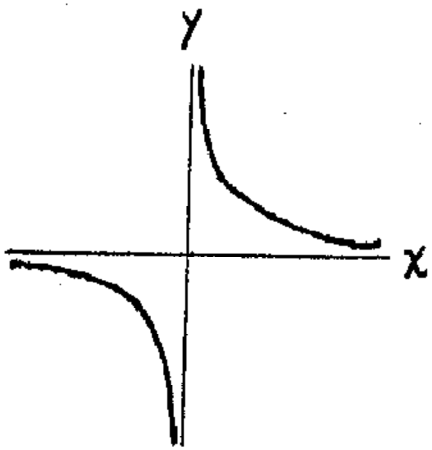
What properties of a function cause "breach" or "holes" in the graph? (so, it does not "continue")

Definition: A function f is continuous at $x = c$ provided all three conditions are satisfied.

1. $f(x)$ is defined [f exists at $x = c$]
2. $\lim_{x \to c} f(x)$ exists [equal a real number]
3. $\lim_{x \to c} f(x) = f(c)$

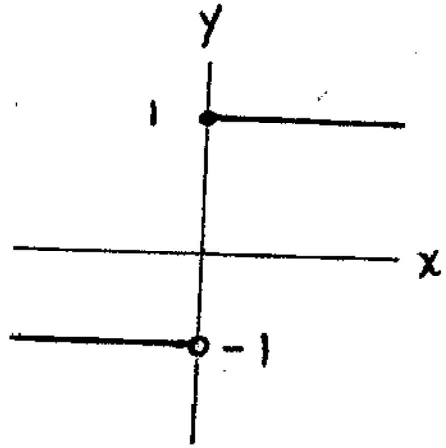
If not, the f is discontinuous at $x = c$

SOME NON-EXAMPLES :



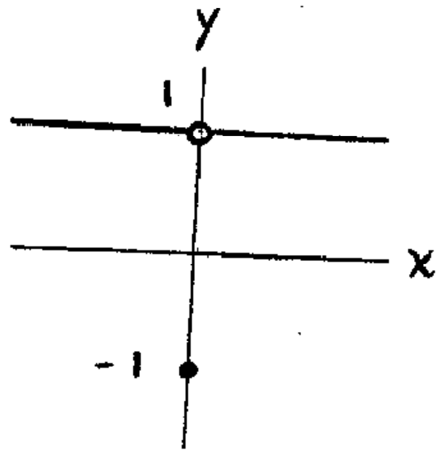
$$f(x) = \frac{1}{x}$$

$f(0)$ NOT DEFINED



$$f(x) = \begin{cases} 1, & x \geq 0 \\ -1, & x < 0 \end{cases}$$

$f(0)$ DEFINED, BUT $\lim_{x \to 0} f(x)$ DNE



$$f(x) = \begin{cases} 1, & x \neq 0 \\ -1, & x = 0 \end{cases}$$

$f(0)$ DEFINED AND $\lim_{x \to 0} f(x)$ EXISTS, BUT THEY'RE NOT THE SAME

Intuitively, $f(x)$ is continuous at $x = a$ if the Graph of $f(x)$ does not break at $x = a$

If $f(x)$ is not continuous at $x = a$ (i.e. if the Graph of $f(x)$ does break at $x = a$), then $x = a$ is a discontinuity of $f(x)$

Note: If $x = a$ is an endpoint for the Domain of $f(x)$, then $\lim_{x\to a} f(x)$ in the definition is replaced by the appropriate one-sided limit, e.g.

$$f(x) = \sqrt{x}$$

Is defined on $[0, \infty)$ and is continuous at $x = 0$ because $f(0) = 0$ and

$$\lim_{x\to 0^+} f(x) = 0$$

Function Continuous on an Interval

- f is continuous on (a, b) or $(-\infty, \infty)$ if f is continuous at each $x = c$ in the interval.

[two-sided limits are possible here at each $x = c$]

What about f defined on $[a, b]$ [No two-sided limits at a or b]

Definition: at $x = c$ f is continuous from the left,

if $\lim_{x \to c^-} f(x) = f(c)$

from the right if

if $\lim_{x \to c^+} f(x) = f(c)$

Definition: f is continuous on $[a, b]$, if

1. f is continuous on (a, b)
2. f is continuous "from the right" at a
3. f is continuous "from the left" at b

Properties and combinations of continuous functions

Recall: If $p(x)$ is a polynomial function, then $\lim_{x\to c} p(x) = p(c)$

So, every polynomial function is continuous everywhere.

Suppose, f, g are continuous at $x = c$

Theorem: $f + g$; $f - g$; $f \cdot g$ are all also continuous at $x = c$

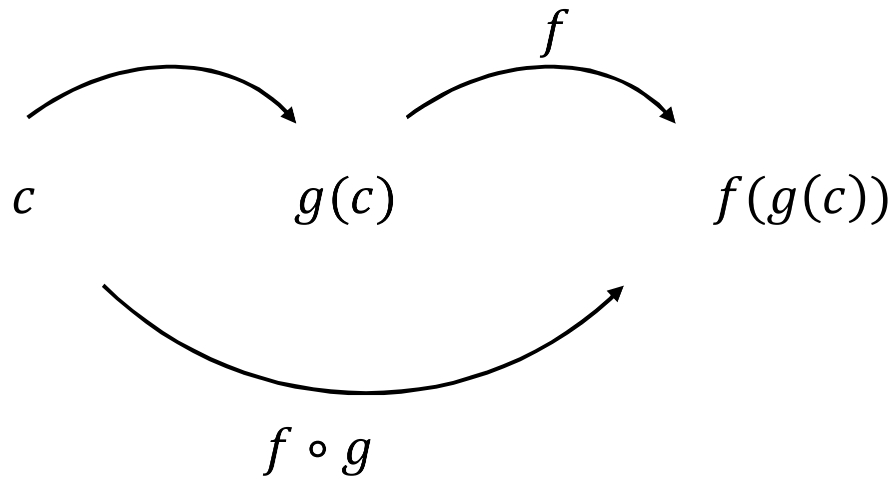
$\frac{f}{g}$ is continuous at $x = c$ provided $g(c) \neq 0$

[otherwise, $\frac{f}{g}$ is discontinuous at $x = c$]

So, every rational function is continuous at every point where the bottom is not zero.

Lastly: “the composition of continuous functions is also continuous”

Theorem: If g is continuous at c and f is continuous at $g(c)$ then $f \circ g$ is continuous at c



The Intermediate Value Theorem and Approximating Roots: $\boldsymbol{f(x) = 0}$

Intermediate Value Theorem

If f is continuous on $[a, b]$ and c is between $f(a)$ and $f(b)$, or equal to one of them, then there is at least one value of x in $[a, b]$ such that $f(x) = c$

Theorem: If f is continuous on $[a, b]$ and $f(a), f(b)$ are non zero with opposite signs, then there is at least one "solution" of $f(x) = 0$ in (a, b)