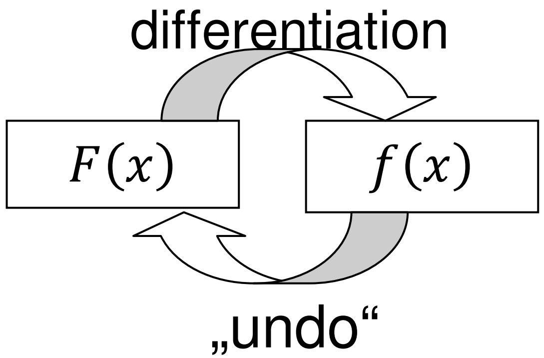
The Integral of a Function. The Indefinite Integral

- Undoing a derivative: Antiderivative=Indefinite Integral

Definition: A function $F(x)$ is called an antiderivative of a function $f(x)$ on same interval $I = [a, b]$, if

$$F'(x) = f(x)$$
$$for\ all\ x\ in\ I$$



Note: Unlike derivatives, anti-derivatives are note unique

$$\frac{d}{dx}\left[\frac{1}{3}x^3\right] = x^2 = f(x)$$

But also for any constant c

$$\frac{d}{dx}\left[\frac{1}{3}x^3 + c\right] = x^2 = f(x)$$

because

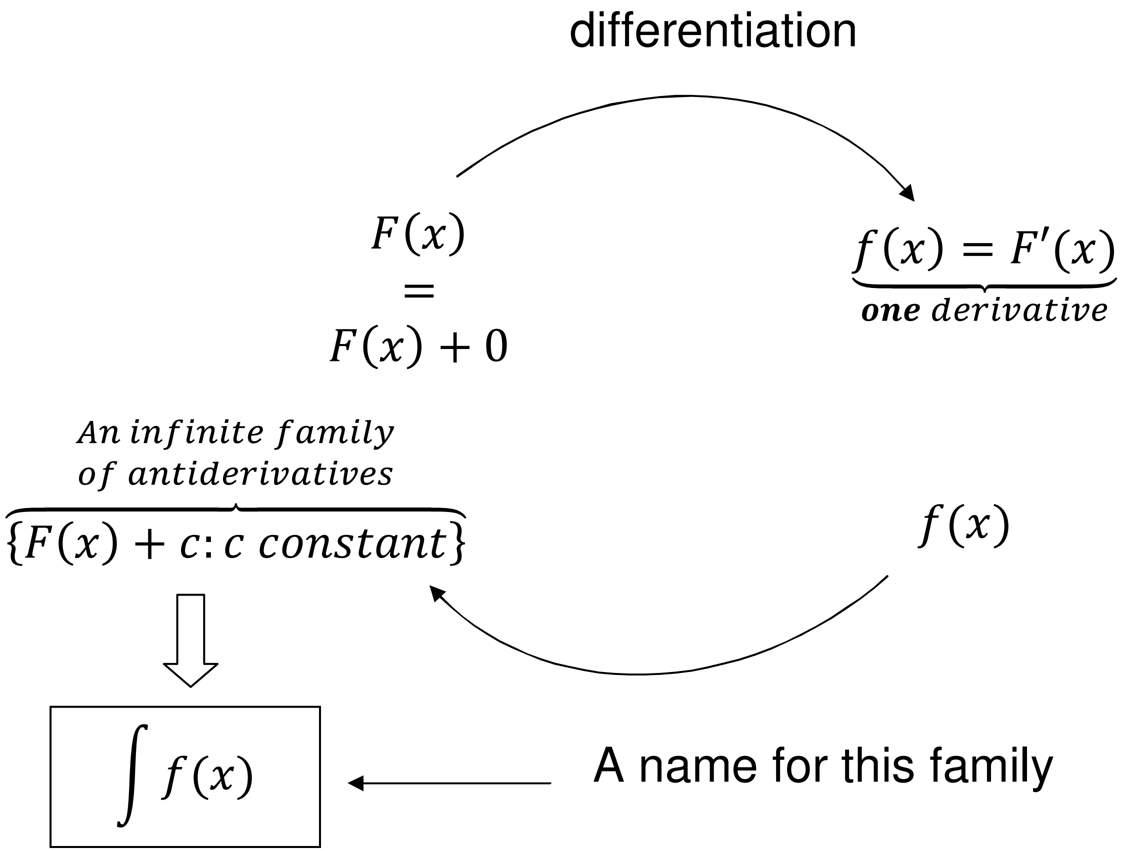
$$\frac{d}{dx}[c] = 0$$

Theorem:

If $F(x)$ is any antiderivative of $f(x)$ on I, Then so is $F(x) + c \leftarrow any\ constant$

Every antiderivative of $f(x)$ on I has the form $F(x) + c$ for some c

- Differentiation produces **one** derivative
- Antidifferentiation produces **an infinite family** of antiderivatives


differentiation
$F(x)$
$=$
$F(x) + 0$
$\underbrace{f(x) = F'(x)}_{\textbf{one}\ derivative}$
$\overbrace{\{F(x) + c : c\ constant\}}^{An\ infinite\ family\ of\ antiderivatives}$
$f(x)$
$\int f(x)$
A name for this family

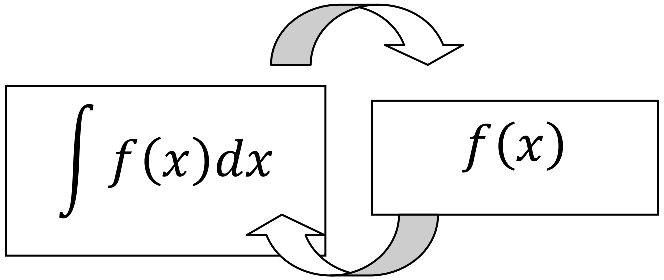
$$\underbrace{\int f(x)}_{\text{The indefinite integral of } f(x)} = F(x) + c$$

- $\int$ – the integral sign [elongated “S”]
- $f(x)$ - the integrand
- dx - indicates the independent variable
- c - constant of integration
- $F(x) + c$ - one of many antiderivative of $f(x)$

The Indefinite Integral of $f(x)$ represents the entire family of all antiderivatives of $f(x)$

So,

Differentiation



Antidifferentiation

[indefinite Integration]

$$\frac{d}{dx}\left[\int f(x)dx\right] = f(x)$$

Note: Sometimes we write:

$$\int 1dx \ or \int dx$$

$$\int \frac{1}{x^2}dx \ or \int \frac{dx}{x^2}$$

Finding anti-derivatives:

(1) Use derivatives we know to build a table

Derivative	Corresponding anti-derivative
$\frac{d}{dx}c = 0$	$\int dx = x + c$
$\frac{d}{dx}\left[\frac{x^{r+1}}{r+1}\right] = x^r$ where $r \neq -1$	$\int x^r\, dx = \left[\frac{x^{r+1}}{r+1}\right] + c$ "Add 1 to the power and divide by this new power"
$\frac{d}{dx}[sinx] = cosx$	$\int cosxdx = sinx + c$
$\frac{d}{dx}[cosx] = -sinx$	$\int sinxdx = -\, cosx + c$
$\frac{d}{dx}[tanx] = \frac{1}{cos^2x}$	$\int \frac{1}{cos^2x}dx = tanx + c$

$\frac{d}{dx}[cotx] = -\frac{1}{sin^2x}$	$\int \frac{1}{sin^2x}dx = -\,cotx + c$
$\frac{d}{dx}[arcsinx] = \frac{1}{\sqrt{1-x^2}}$	$\int \frac{1}{\sqrt{1-x^2}}dx = -\,arcsinx + c$
$\frac{d}{dx}[arctanx] = \frac{1}{1+x^2}$	$\int \frac{1}{1+x^2}dx = arctanx + c$
$\frac{d}{dx}[arccotx] = -\frac{1}{1+x^2}$	$\int \frac{1}{1+x^2}dx = -\,arctanx + c$
$\frac{d}{dx}[e^x] = e^x$	$\int e^x dx = e^x + c$
$\frac{d}{dx}[a^x] = a^x lna$	$\int a^x dx = \frac{a^x}{lna} + c$
$\frac{d}{dx}[lnx] = \frac{1}{x}$	$\int \frac{1}{x}dx = ln\,x + c$
$\frac{d}{dx}[\log_a x] = \frac{1}{xlna}$	$\int \frac{1}{x}dx = ln\,x + c$

$\int lnx dx = x(lnx) - x + c$
$\int \log_a x\, dx = \frac{1}{lna}(x \cdot (lnx) - x) + c$

(2) Some Properties on Indefinite Integrals: c a real number

$$\int cf(x)dx = c\int f(x)dx$$

$$\int [f(x) + g(x)]dx = \int f(x)dx + \int g(x)dx$$

$$\int [f(x) - g(x)]dx = \int f(x)dx - \int g(x)dx$$

All applied earlier for limits + derivatives

Examples:

Do not write

$$\int 2x dx = 2\int x dx = 2\left(\frac{x^2}{2} + c\right) = x^2 + 2c = x^2 + c$$

$$\int (1+x)dx = \int 1dx + \int x dx = (x + c_1) + \left(\frac{x^2}{2} + c_2\right) = x + \frac{x^2}{2} + c$$

Note on constant of integration

- Do not forget constants of integrations
- Do not introduce them too soon
- Combine multiple constant into one c

What integration technique so far?

(1) Use (create) a table
(2) Rewrite an integrand (in order to use the table)

Examples:

$$\int 2 \cdot x^2 dx = 2 \cdot \int x^2 dx = \frac{2}{3} x^3 + c$$

$$\int (x^2 + 3sinx)dx = \int x^2 dx + \int 3sinxdx = \frac{x^3}{3} - 3cosx + c$$

The Indefinite Integration by Parts

$$\int f(x) \cdot g(x)\, dx = ?$$

Recall the product rule for derivatives $u = u(x)$, $v = v(x)$

$$[u(x) \cdot v(x)]' = u'(x) \cdot v(x) + u(x) \cdot v'(x)$$

$$u'(x) \cdot v(x) = [u(x) \cdot v(x)]' - u(x) \cdot v'(x)$$

Integrate both sides

$$\int u'(x) \cdot v(x)\, dx = \int [u(x) \cdot v(x)]' dx - \int u(x) \cdot v'(x)\, dx$$

$$\int u'(x) \cdot v(x) = uv - \int u(x) \cdot v'(x)\, dx$$

Shorthand notation: The integration by part formula

$$\int v du = uv - \int u dv$$

Note: Identify something to call u. The rest is dv. Compute dv by differentiation and v by integration. Plug in the integration by parts formula and hope that the new integral is easier than the original one.

Generally try to choose u to be something that simplifies when you differentiate it.

Examples:

1.

$$\int 2xe^x dx$$

How to choose u and v?

$u(x) = 2x \quad v(x) = e^x$

$U(x)$ and $v'(x)$ are easy to find: $U(x) = x^2$ und $v'(x) = e^x$

But we cannot find the indefinite Integral of the product $U(x)v'(x) = x^2 \cdot e^x$

Then:

$u(x) = e^x \quad v(x) = 2x$

$U(x) = e^x$ and $v'(x) = 2$

$\int U(x)v'(x)dx = 2e^x$

$$\int e^x 2x dx = 2xe^x - 2\int e^x dx = e^x(2x - 2) + c$$

2.

$$\int e^x x^2 dx$$

$$u'(x) = e^x \quad v(x) = x^2$$

$$u(x) = e^x \; v'(x) = 2x$$

$$\int e^x x^2 dx = e^x x^2 - \int e^x 2x dx = e^x x^2 - 2xe^x + 2\int e^x dx = e^x(x^2 - 2x + 2) + c$$

3.

$$\int cosx \cdot sinxdx$$

$$\int cosx \cdot sinxdx = sinx \cdot sinx - \int sinx \cdot cosxdx$$

$$2\int cosx \cdot sinxdx = sin^2 x$$

$$\int cosx \cdot sinxdx = \frac{1}{2} sin^2 x$$

The Indefinite Integration by Substitution

Idea: Suppose $F' = f$ and g' exists

Chain rule:

$$F'(g(x)) = \underbrace{F'(g(x))}_{outside} \cdot \underbrace{g'(x)}_{inside}$$

$$\int F'(g(x)) \cdot g'(x)\, dx = F(g(x)) + c$$

So,

$$\int f(g(x)) \cdot g'(x)\, dx = F(g(x)) + c$$

Let $u = g(x)$, then $\frac{du}{dx} = g'(x), du = g'(x)dx$

$$\int f(u)\, du = F(u) + c$$

Substitution of u for $g(x)$ makes (when it works!) integration easier.

Straightforward Substitution

- Always consider "Substitution" first
- If on substitution fails, try another one!

Always make a total change from x to u! Never mix variables!

Substitution technique: Find something in the integrand to call u to simplifies the appearance of the integral and whose $du = \frac{du}{dx}dx$ is also present as a factor

Examples:

$$\int \sqrt{\underbrace{1+x}_{t}}\,dx = \int \sqrt{t}\,dt = \frac{2}{3}t^{\frac{3}{2}} = \frac{2}{3}(1+x)^{\frac{3}{2}}$$

Exercises:

Find the Indefinites Integral of the following functions by Substitution

1.

$$f(x) = e^{2x}$$

2.

$$f(x) = (x + 1)^2$$

3.

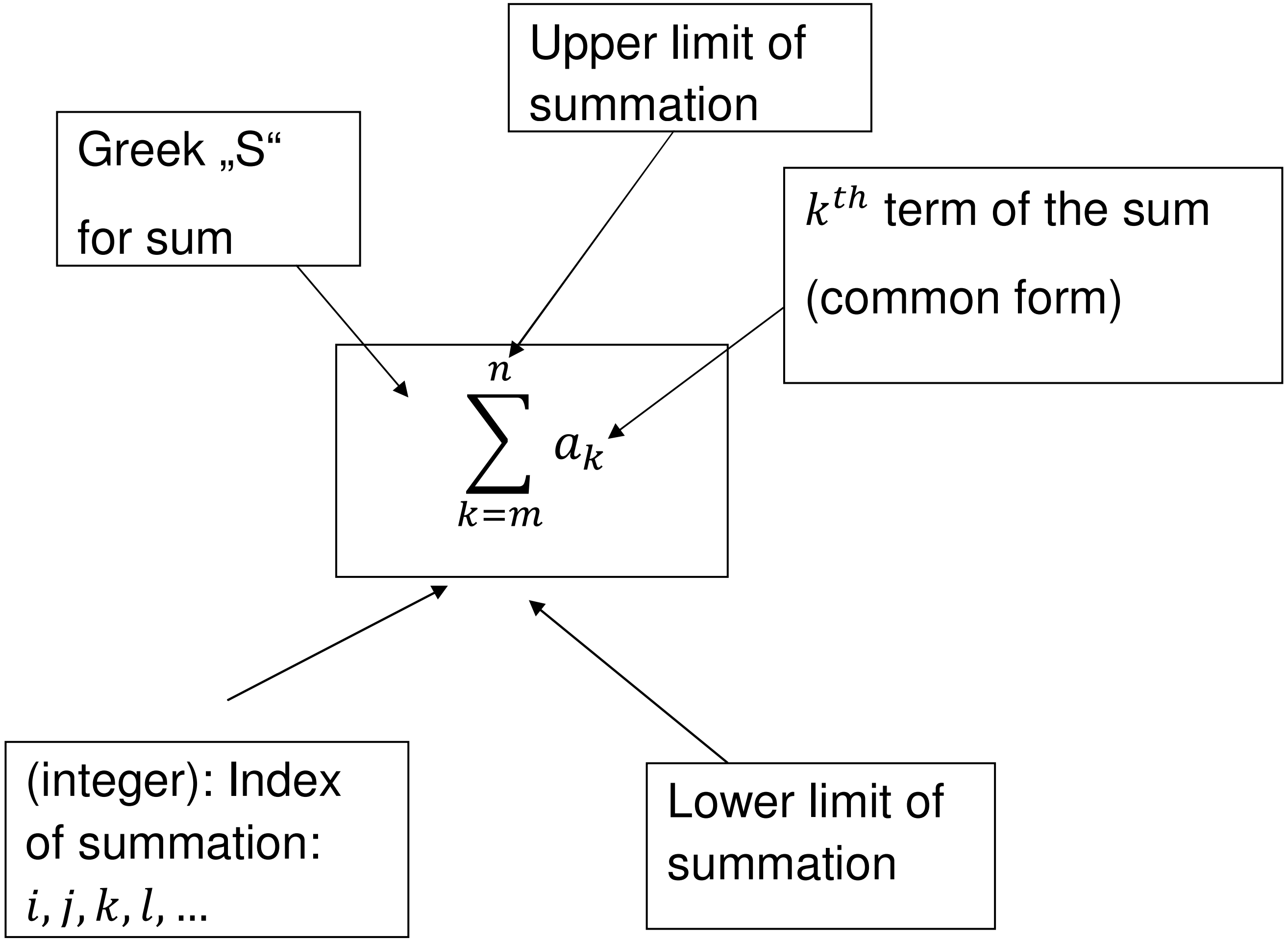
$$f(x) = xln(x^2)$$

Solution:

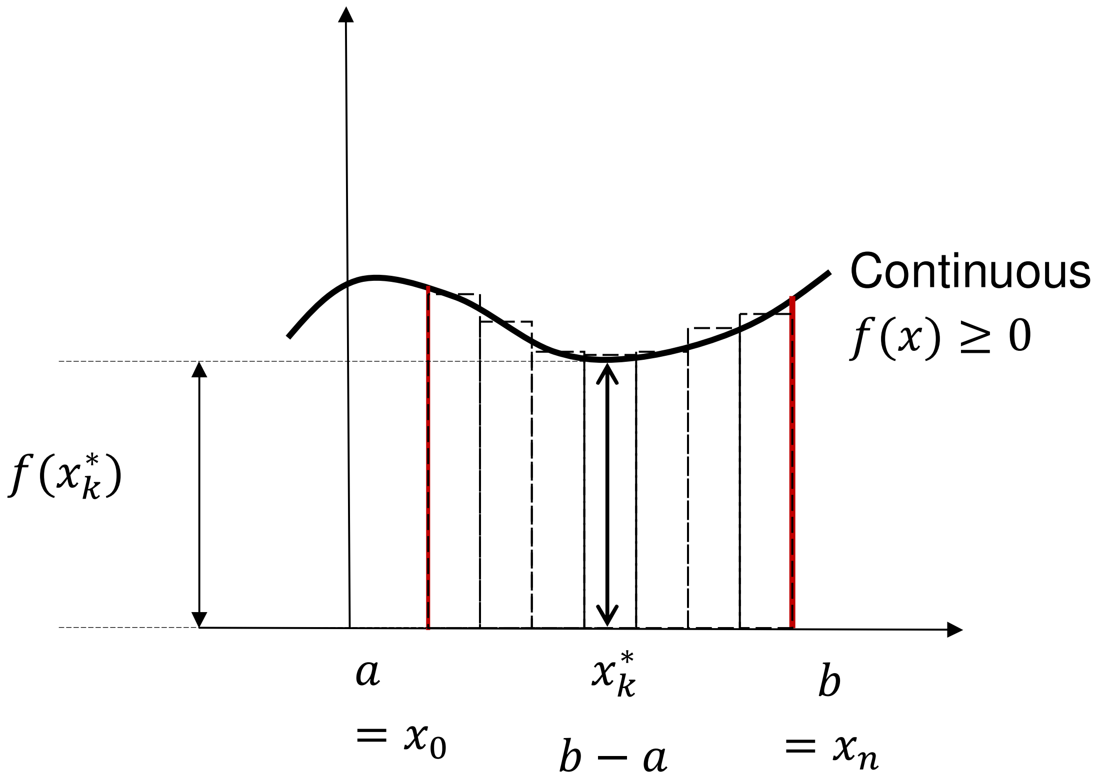
function	substitution	Integral
$f(x) = e^{2x}$	$t = 2x$	$F(x) = \frac{1}{2}e^{2x} + c$
$f(x) = (x+1)^2$	$t = x + 1$	$F(x) = \frac{(x+1)^3}{3} + c$
$f(x) = xln(x^2)$	$t = x^2$	$F(x) = \frac{1}{2}x^2 ln(x^2) - \frac{1}{2}x^2$

Area Defined as a Limit

The Sigma $\sum$ shorthand for sums



Definition of Area "under a Curve"



- Partition into n **equal** subintervals
- Each width $= \frac{1}{n}(b-a) = \Delta x$
- Choose any point in each interval to calculate rectangle heights

$$\begin{bmatrix} Area\ under \\ Curve \end{bmatrix} \approx \sum_{k=1}^{n} \left[\underbrace{f(x_k^*)\Delta x}_{\substack{Area\ of\ one \\ rectangle}} \right]$$

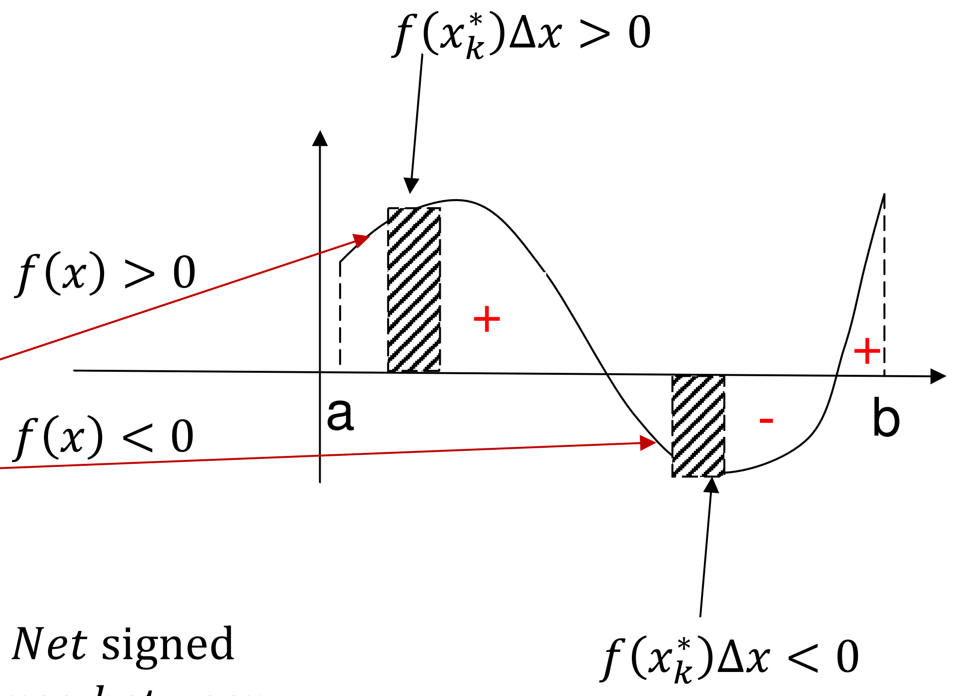
Definition: If f is continuous on $[a,b]$ $f(x) \geq 0$ on $[a,b]$

Then

$$\begin{bmatrix} Area\ under \\ y = f(x) \\ over\ [a,b] \end{bmatrix} = \lim_{n \to \infty} \sum_{k=1}^{n} f(x_k^*)\Delta x$$

Net Area

Definition: net „signed area“



Not!

If f is continuous on $[a, b]$ $f(x) \geq 0$ Then

$$\lim_{n\to\infty}\left[\sum_{k=1}^{n} f(x_k^*)\Delta x\right] = \begin{matrix} Net\ \text{signed} \\ Area\ between \\ y = f(x)\ and\ [a, b] \end{matrix}$$

Approximating Area Numerically

For large n

$$\lim_{n\to\infty}\sum_{k=1}^{n} f(x_k^*)\Delta x \approx \sum_{n\to\infty}^{n} f(x_k^*)\Delta x$$

The Definite Integral

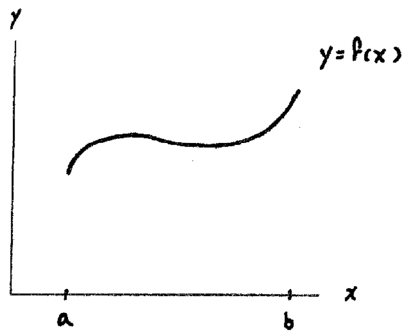
The Definite Integral Defined

- Extend our “Net Area” limit

$$\lim_{n\to\infty} \sum_{k=1}^{n} f(x_k^*)\Delta x$$

Continuous function

Equal lenght subinterval



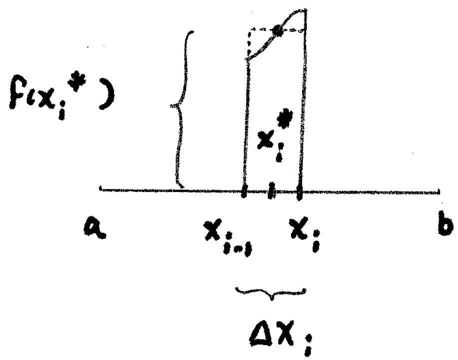
To compute the area under the graph of $f(x)$ and above the interval $[a, b]$ we proceed as follows:

1. Subdivide the interval $[a, b]$ into n **unequal** subintervals with endpoints: $a = x_0 < x_1 < x_2 < \cdots < x_{n-2} < x_{n-1} < x_n = b$

For each $k = 1,2, \dots n-1, n$ let $\Delta x_k = x_k - x_{k-1} = lenght\ of\ [x_{k-1}, x_k]$

Note: The largest of the Δx_k will be denoted Δx_{max}

2. Inside each $[x_{k-1}, x_k]$select a point x_k^*, evaluate $f(x_1^*), f(x_2^*), \dots, f(x_{n-1}^*), f(x_n^*)$ and compute $f(x_1^*)\Delta x_1, f(x_2^*)\Delta x_2, \dots, f(x_{n-1}^*)\Delta x_{n-1}, f(x_n^*)\Delta x_n$



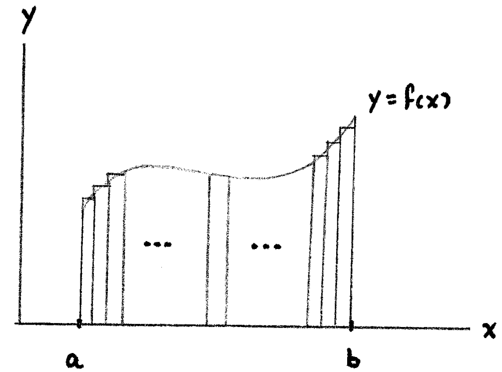
3. Form the Riemann Sum Approximation

$$f(x_1^*)\Delta x_1 + f(x_2^*)\Delta x_2 + \cdots + f(x_{n-1}^*)\Delta x_{n-1} + f(x_n^*)\Delta x_n = \sum_{k=1}^{n} f(x_k^*)\Delta x_k$$

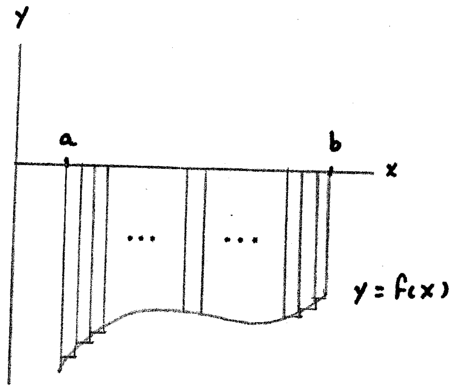
4. Repeat Step 1-3 over and over with finer and finer subdivision of $[a, b]$ (i.e. smaller and smaller Δx_{max} and take a limit

$$\lim_{\Delta x_{max} \to 0} \sum_{k=1}^{0} f(x_k^*)\Delta x_k$$

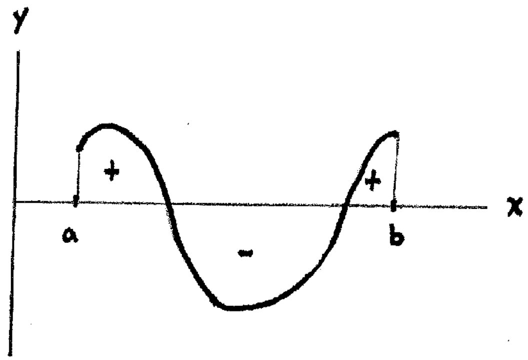
Partition in **equal** subinterval: $n \to \infty$ means $\Delta x \to 0$ guaranties each width shrinks
Partition in **unequal** subinterval: $max\Delta x_k \to 0$ guaranties each width shrinks



Notice that if $f(x) \leq 0$ on $[a, b]$, then the result of this procedure will be minus the area between the graph of $f(x)$ and $[a, b]$.



If $f(x)$ takes both positive and negative value on $[a, b]$, then the procedure yield the net signed area between the graph of $f(x)$ and the interval $[a, b]$



Definite Integral: Definition

a. f is integrable on $[a, b]$ if

$$\lim_{max\Delta x_k \to 0} \sum_{k=1}^{n} f(x_k^*)\Delta x_k$$

exists and does not depend on

- the choice of partition
- or the choice of $x_k{}^*$ point

b. If f is integrable, then the limit

$$\lim_{max\Delta x_k \to 0} \sum_{k=1}^{n} f(x_k^*)\Delta x_k$$

is called the **Definite Integral** of $f(x)$ over $[a, b]$ [or from a to b] and is denoted

$$\int_a^b f(x)dx$$

$$\int_a^b f(x)dx$$

a: lower limit of integration

b: upper limit of integration

Be careful not to confuse $\int_a^b f(x)dx$ and $\int f(x)dx$. They are **entirely different types** of things. The first is a **number**, the second is a collection of functions.

Notation:

$$\Delta \to d$$

$$\Delta x \to dx$$

$$\sum \quad \to \int$$

The definite Integral of a continuous Function = Net "Area" under a curve

Theorem: If f is continuous on $[a, b]$

Then f is integrable on $[a, b]$

And

$$\begin{matrix} Net \ \pm Area \\ between\ the \\ graph\ of\ f \\ and\ [a, b] \end{matrix} = \int_a^b f(x)dx$$

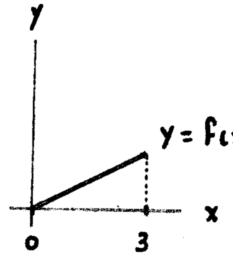
Notation:

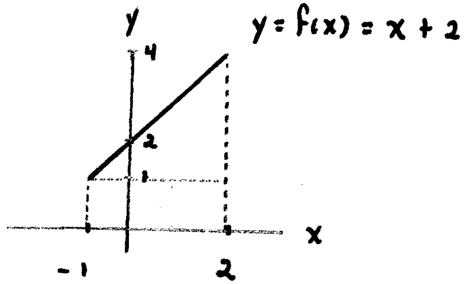
$$\int_{x=a}^{x=b} [integrand]dx$$

We will need methods for evaluating the number $\int_a^b f(x)dx$ other than computing the limit that defines them.

Some methods generally involve antidifferentiation, but some definite integrals can be evaluated by thinking of them as area, e.g.

Definite Integrals using geometry





$$\int_0^3 \frac{1}{2} x dx = \frac{1}{2} 3 \left(\frac{1}{2} \cdot 3\right) = \frac{9}{4}$$

$$\int_{-1}^{2} (x+2)dx = \frac{1}{2} 3 \cdot 3 + 3 \cdot 1 = \frac{9}{2} + 3 = \frac{15}{2}$$

Finding Definite Integrals: A new definition and properties:

a. If a is in Domain of f, define

$$\int_a^a f(x)dx = 0$$

b. If f is integrable on $[a, b]$, define

$$\int_b^a f(x)dx = -\int_a^b f(x)dx$$

$$\int_a^b [cf(x)]dx = c\int_a^b f(x)dx$$

$$\int_a^b [f(x) + g(x)]dx = \int_a^b f(x)dx + \int_a^b g(x)dx$$

$$\int_a^b [f(x) - g(x)]dx = \int_a^b f(x)dx - \int_a^b g(x)dx$$

Theorem: If f is integrable on any closed Interval containing a, b, c

Then

$$\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx$$

No matter, how a, b, c are ordered!

Theorem: Suppose, f, g integrable on $[a, b]$

a. If $f(x) \geq 0$ for all x in $[a, b]$, Then

$$\int_a^b f(x)dx \geq 0$$

b. If $f(x) \geq g(x)$ for all x in $[a, b]$, Then

$$\int_a^b f(x)dx \geq \int_a^b g(x)dx$$

The Fundamental Theorem of Calculus

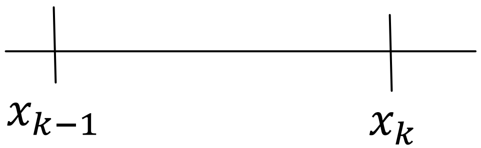
There are two parts to this.

The Fundamental Theorem of Calculus, Part I

Development:

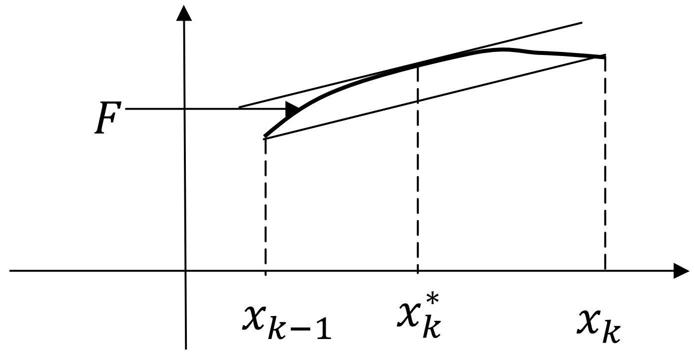
Suppose: f is continuous on $[a, b]$ and $F' = f$ [F differentiable means Fcontinuous]

Partition



$$x_k - x_{k-1} = \Delta x_k$$

on each interval:



The Mean Value Theorem for derivatives applied to F on each interval

$$F'(x_k^*) = \frac{F(x_k) - F(x_{k-1})}{x_k - x_{k-1}}; \quad F'(x_k^*)(x_k - x_{k-1}) = F(x_k) - F(x_{k-1})$$

$$f(x_k^*)\Delta x_k = F(x_k) - F(x_{k-1})$$

$$f(x_1^*)\Delta x_1 = F(x_1) - F(a)$$

$$f(x_2^*)\Delta x_2 = F(x_2) - F(x_1)$$

$$\vdots \qquad \vdots \qquad \vdots$$

$$f(x_n^*)\Delta x_n = F(b) - F(x_{n-1})$$

$$\sum_{k=1}^{n} f(x_k^*)\Delta x_k = F(b) - F(a)$$

Taking a limit as $max\Delta x_k \to 0$ give us the definite Integral

FTC, Part I

If f is continuous on $[a, b]$ and $F(x)$ is any anti-derivative for $f(x)$ on $[a, b]$.

then

$$\int_a^b f(x)dx = F(x)|_a^b = \underbrace{F(b)}_{upper} - \underbrace{F(a)}_{lower}$$

Notice. If F is any antiderivative of f,

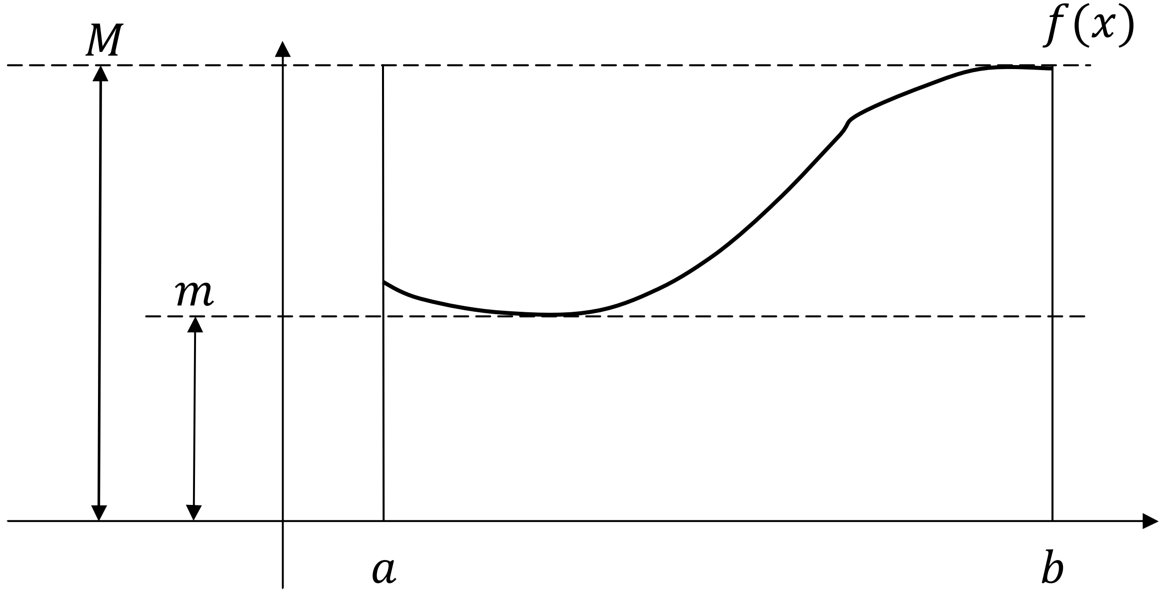
$$\int_a^b f(x)dx = [F(x) + c]_a^b = [F(b) + c] - [F(a) + c] = F(b) - F(a)$$

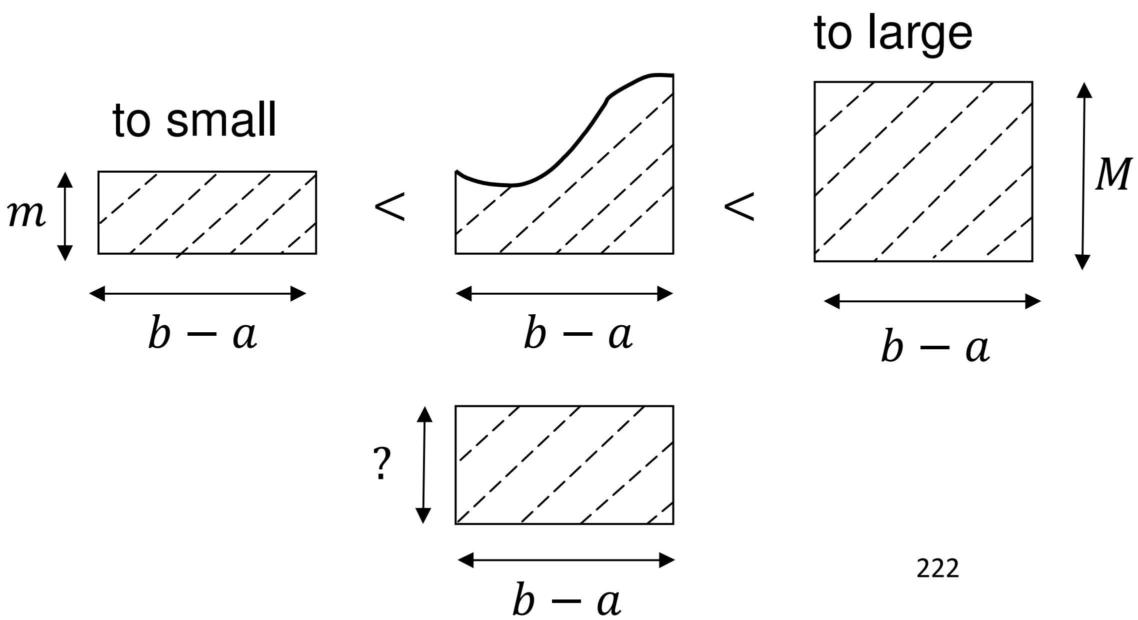
So, we can always omit writing c here. Thus

$$\int_a^b f(x)dx = F(x)|_a^b$$

The Mean Value Theorem for Integrals

Idea:





If f is continuous on $[a,b]$, then there is at least one point x^* in $[a,b]$ such, that

$$\int_a^b f(x)dx = \underbrace{f(x^*)}_{\substack{mean \\ (average) \\ height}} (b-a)$$

Proof: By the Extreme Value Theorem f assumes both a $max\ M$ and $\min\ m$ on $[a, b]$.

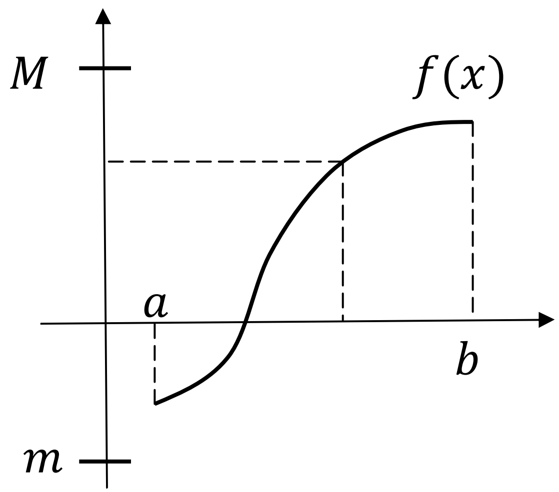
So, $m \leq f(x) \leq M$

Thus

$$\int_a^b m\,dx \leq \int_a^b f(x)dx \leq \int_a^b M\,dx$$

$$m(b-a) \leq \int_a^b f(x)dx \leq M(b-a)$$

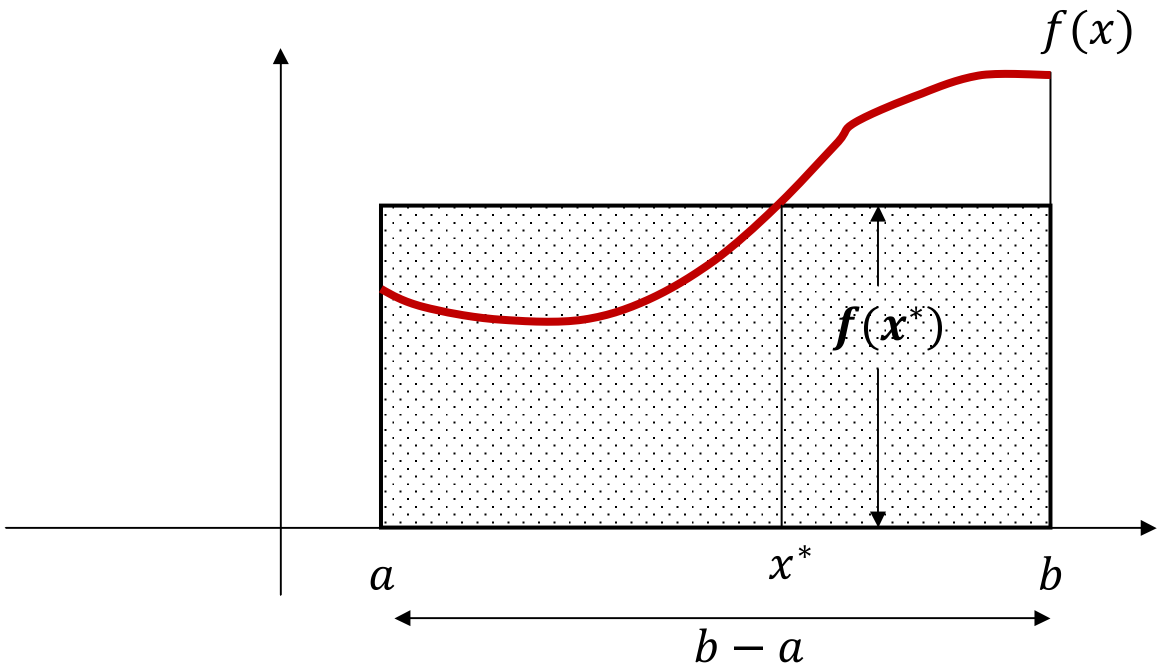
$$m \leq \underbrace{\frac{1}{b-a}\int_a^b f(x)dx}_{A\ number!} \leq M$$



By Intermediate Value Theorem for some x^* in $[a, b]$

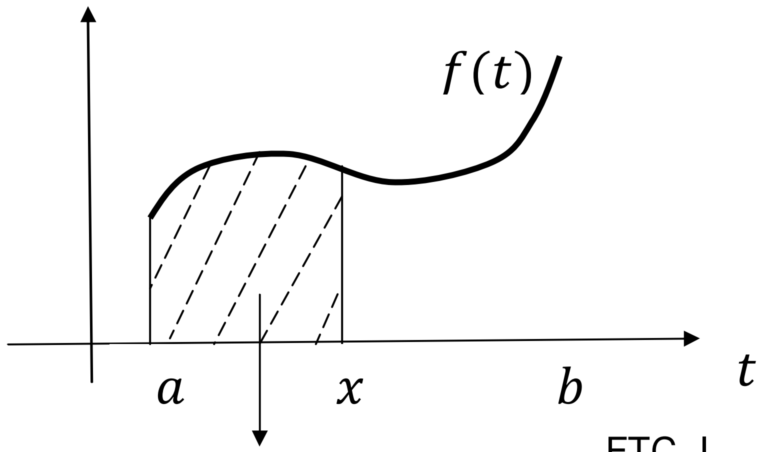
$$f(x^*) = \frac{1}{b-a}\int_a^b f(x)dx$$

Thus



$$\int_a^b f(x)dx = f(x^*)(b-a)$$

The Fundamental Theorem of Calculus, Part II



$F' = f$

FTC, I

$$\int_a^x f(t)dt = F(x) - F(a)$$

$$\int_a^x f(t)dt = F(x) - F(a)$$

$$\frac{d}{dx}\left[\int_a^x f(t)dt\right] = \underbrace{F'(x)}_{f(x)} - \underbrace{F'(a)}_{=0} = F'(x)$$

The fundamental theorem of calculus says that is always true:

If f is continuous on the Interval I,

Then f has an antiderivative on I

If a is in I

Then

$$F(x) = \int_a^x f(t)dt$$

is one such antiderivative for $f(x)$ meaning

$$\frac{d}{dx}\left[\int_a^x f(t)dt\right] = f(x)$$

Differentiation and Integration are Inverse Processes:

FTC, PartI

$$\int_a^x f'(t)dt = f(x) - f(a)$$

"Integral of derivative recovers original function"

FTC, PartII

$$\frac{d}{dx}\left[\int_a^x f(t)dt\right] = f(x)$$

"Derivative of integral recovers original function".

Definite and Indefinite Integrals Related:

$$\int f(x)dx$$

Is a function in x,

$$\int_a^b f(x)dx$$

Is a number – no x involved!

So, the variable of integration in a definite integral doesn't matter: The name of the variable is irrelevant. For this reason the variable in a definite integral is often referred to as dummy variable, place holder.

$$\int_a^b f(x)dx = \int_a^b f(t)dt = \int_a^b f(y)dy$$

Some Examples:

1.

$$\int_4^4 2xdx = x^2|_4^4 = 4^2 - 4^2 = 0$$

2.

$$\int_1^2 2xdx = x^2|_1^2 = 2^2 - 1^2 = 3$$

$$-\int_2^1 2xdx = -x^2|_2^1 = -1^2 + 2^2 = 3$$

3.

$$\int_1^4 2xdx = x^2|_1^4 = 4^2 - 1^2 = 15$$

$$\int_1^2 2xdx + \int_2^4 2xdx = x^2|_1^2 + x^2|_2^4 = 2^2 - 1^2 + 4^2 - 2^2 = 15$$

Definite Integration by Substitution.

Extending the Substitution Method of Integration to definite Integrals

to evaluate the number

$$\int_a^b f\left(g(x)\right)g'(x)dx \qquad \begin{matrix} g' \textit{ continuous on } [a,b] \\ f \textit{ continuous where } g \textit{ exists on } [a,b] \end{matrix}$$

Substitution:

$$u = g(x)$$

$$du = g'(x)dx$$

Change x - limits to u -limits with the substitution:

$$u(a) = g(a)$$
$$u(b) = g(b)$$

To get

$$\int_{g(a)}^{g(b)} f\,(u)du$$

Examples:

1. Find

$$\int_{-1}^{1} e^{2x}$$

1. x substitution of x: $t(x) = 2x = t$ $\frac{dt}{dx} = 2$ $dx = \frac{1}{2}dt$

2. limits substitution:

lower limit: $t(-1) = -2$

upper limit: $t(1) = 2$

$$\frac{1}{2}\int_{-2}^{2} e^{t} = \frac{1}{2}(e^{2} - e^{-2})$$

2. Find:

$$\int_{1}^{2} 2xlnx^2$$

1. x substitution: $t(x) = x^2 = t \quad \frac{dt}{dx} = 2x \quad dx = \frac{1}{2x}dt$

2. limits substitution:

lower limit: $t(1) = 1$

upper limit: $t(2) = 4$

$$\int_{1}^{4} lntdt = tlnt - t|_1^4 = (4ln4 - 4) - (ln1 - 1) = 4ln4 - ln1 - 3$$

The Definite Integral Applied

Total Area

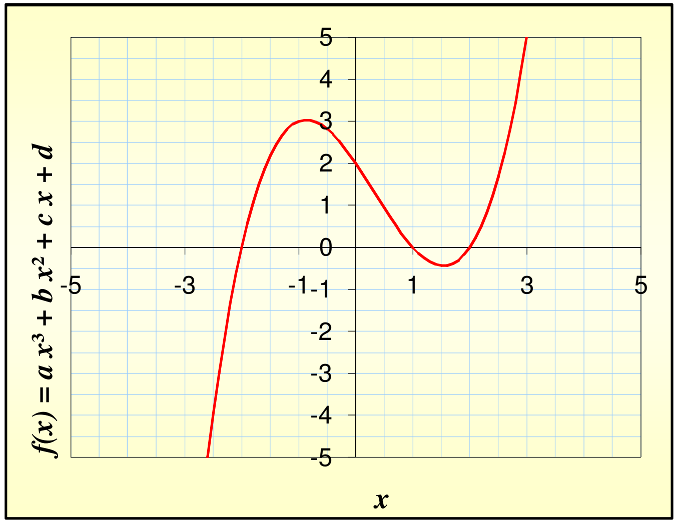
Although

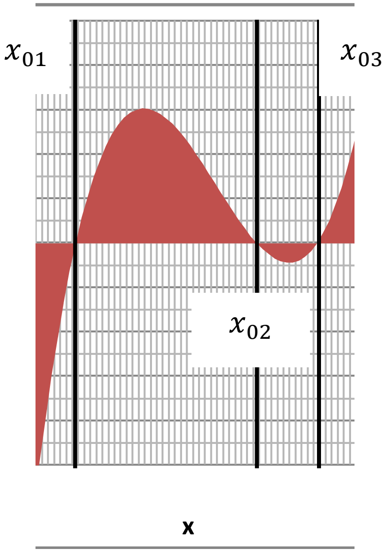
$$\int_a^b f\,(x)dx - "net\ area"$$

We can find that

$$\begin{bmatrix} total \\ Area \end{bmatrix} = \int_a^b |f(x)|\, dx$$

Example. Compute the area between $(x) = 0{,}5x^3 - 0{,}5x^2 - 2x + 2$, the x –axis and the lines $x_1 = -2{,}5$ and $x_2 = 2{,}5$:





$x_{01} = -2, x_{02} = 1\ x_{03} = 2$

Nullpoints

$$f(x) = 0{,}5x^3 - 0{,}5x^2 - 2x + 2 = 0{,}5(x+2)(x-1)(x-2)$$

$$F(x) = 0{,}5 \cdot \frac{1}{4}x^4 - 0{,}5\frac{1}{3}x^3 - 2\frac{1}{2}x^2 + 2x = \frac{1}{8}x^4 - \frac{1}{6}x^3 - x^2 + 2x$$

Area

$$\mathcal{F} = \left|\int_{-2,5}^{-2} f(x)\,dx\right| + \int_{-2}^{1} f(x)\,dx + \left|\int_{1}^{2} f(x)\,dx\right| + \int_{2}^{2,5} f(x)\,dx$$

$$\begin{aligned}\mathcal{F} &= |F(-2) - F(-2{,}5)| + F(1) - F(-2) + |F(2) - F(1)| + F(2{,}5) - F(2) = \\ &= |-4{,}67 + 3{,}76| + 0{,}96 - (-4{,}66) + |0{,}67 - 0{,}96| + 1{,}03 - 0{,}67 = \\ &= 0{,}90 + 5{,}625 + 0{,}29 + 0{,}36 = 7{,}175\end{aligned}$$

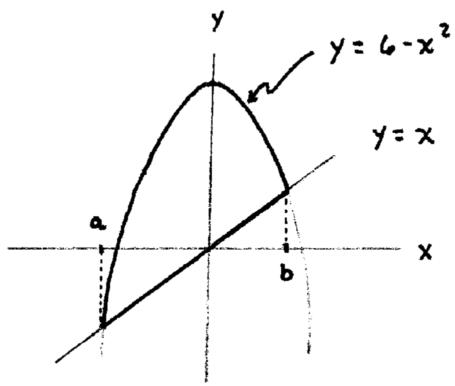
Area between Two Curves [one floor, one ceiling]

$$\begin{bmatrix} Area\ between \\ curves \end{bmatrix} = \int_a^b \begin{bmatrix} \underbrace{f(x)}_{upper} - \underbrace{g(x)}_{lower} \\ one\ ceiling - one\ floor \end{bmatrix} dx$$

Example:

Compute the area of the region between the graphs of $y = x$ and $y = 6 - x^2$.

To identify the top $y = f(x)$ and the bottom $y = g(x)$ and the interval $[a, b]$ we need a sketch.



Intersections:

$$6 - x^2 = x$$

$$x^2 + x - 6 = 0$$

$$(x + 3)(x - 2) = 0$$

$$x = -3, 2$$

$$[a, b] = [-3,2]$$

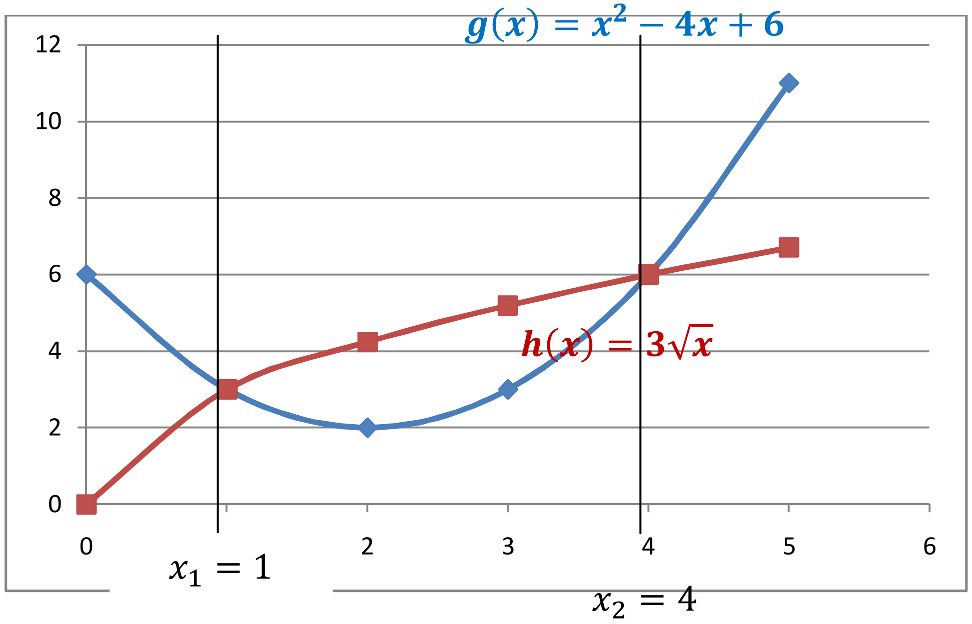
Area:

$$\int_{-3}^{2} \big((6 - x^2) - x\big)\, dx = \int_{-3}^{2} (6 - x^2 - x)\, dx = 6x \,\Big|_{-3}^{2} - \frac{1}{3}x^3 \,\Big|_{-3}^{2} - \frac{1}{2}x^2 \,\Big|_{-3}^{2}$$

$$= 6\big(2 - (-3)\big) - \frac{1}{3}\big(8 - (-27)\big) - \frac{1}{2}(4 - 9) = \frac{125}{6}$$

Compute the area of the region between two graphs

$$g(x) = x^2 - 4x + 6 \text{ and } h(x) = 3\sqrt{x}$$



Intersections:

$$x^2 - 4x + 6 = 3\sqrt{x}$$

$$x_1 = 1,\ x_2 = 4$$

Area:

$$\int_1^4 \left(3\sqrt{x} - x^2 + 4x - 6\right) dx = 2x^{\frac{3}{2}} - \frac{1}{3}x^3 + 2x^2 - 6x \Big|_1^4 =$$

$$= \left(16 - \frac{64}{3} + 32 - 24\right) - \left(2 - \frac{1}{3} + 2 - 6\right) = \frac{8}{3} + \frac{7}{3} = 5$$