The meaning of dx and dy

Definition: Suppose f is differentiable at x

- Define dx an independent variable
- Define $dy = f'(x)dx$,
- Then $\frac{dy}{dx} = \frac{f'(x)dx}{dx} = f'(x)$

$$\left.\begin{aligned} dy &= \text{"the rise"} \\ dx &= \text{"the run"} \end{aligned}\right\} \text{slope of the tangent at } x$$

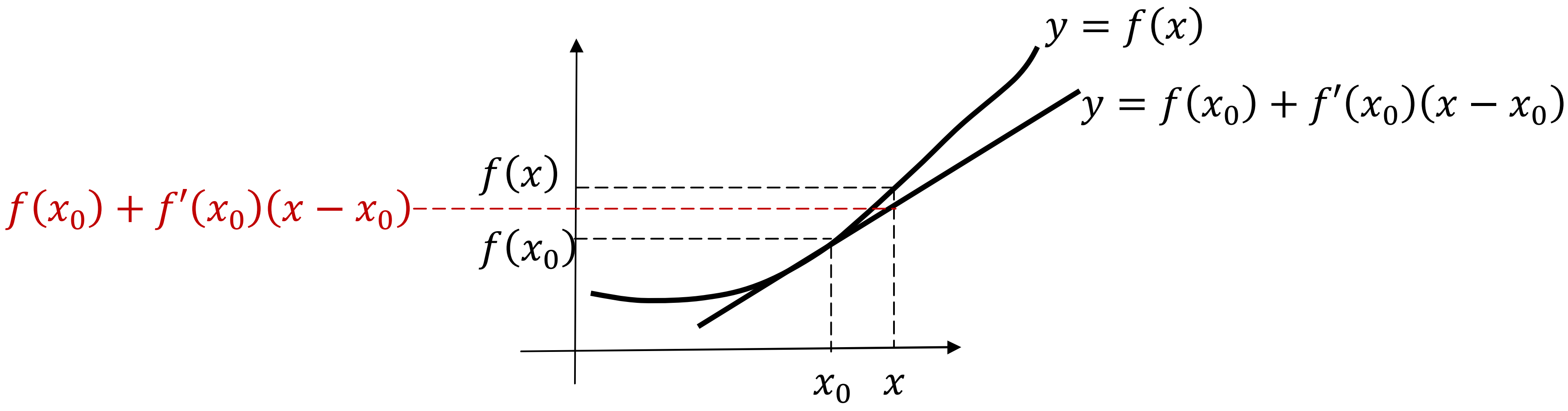
dx, dy - “differentials”

dy: rise of a function

Δy: rise of a tangent

$$\Delta y \approx dy$$

Local linear Approximations of Non-Linear Functions



Tangent line at $(x_0, f(x_0))$

$$y - f(x_0) = f'(x_0)(x - x_0)$$

$$y = f(x_0) + f'(x_0)(x - x_0)$$

For value of x near x_0 then

$$f(x) \approx f(x_0) + \underbrace{f'(x_0)(x - x_0)}_{tangent\ at\ x_0}$$

A local linear approximation of $f(x)$ near x_0

Another way of writing this:

Let $x - x_0 = \Delta x$, so $x = x_0 + \Delta x$

$$f(x_0 + \Delta x) \approx f(x_0) + f'(x_0)\Delta x$$

Finding Limits Using Differentiation: L'Hôpital Rule

Limits of Quotients That Appear to be "Indeterminate": $\frac{0}{0}, \frac{\infty}{\infty}, 0 \cdot \infty, \infty - \infty$

Example:

$$\lim_{x \to 1} \left[\frac{x^2 - 1}{x - 1} \right]$$

has "$\frac{0}{0}$" form.

$$\lim_{x \to 1} \left[\frac{x^2 - 1}{x - 1} \right] = \lim_{x \to 1} \left[\frac{(x - 1)(x + 1)}{x - 1} \right] = \lim_{x \to 1} (x + 1) = 2$$

is doable.

1. Assumption: Suppose

$$\lim_{x \to a} \left[\frac{f(x)}{g(x)} \right]$$

has the $\frac{0}{0}$ form

meaning both

$$\lim_{x \to a} f(x) = 0$$

and

$$\lim_{x \to a} g(x) = 0$$

2. Assumption: Suppose f, g are both differentiable at a

Meaning

$$\left.\begin{matrix} \lim_{x \to a} f(x) = f(a) \\ \lim_{x \to a} g(x) = g(a) \end{matrix}\right\} f(a) = g(a) = 0$$

Observe:

$$\frac{f(x)}{g(x)} = \frac{f(x) - \overbrace{f(a)}^{=0}}{g(x) - \underbrace{g(a)}_{=0}} = \frac{\left[\frac{f(x)-f(a)}{x-a}\right]}{\left[\frac{g(x)-g(a)}{x-a}\right]}$$

Theorem: L'Hôpital Rule (for $\frac{0}{0}$ form)

If f, g are both differentiable on $I, a \in I$ and both

$$\lim_{x\to a} f(x) = 0 \quad and \quad \lim_{x\to a} g(x) = 0$$

Then

$$\lim_{x\to a}\left[\frac{f(x)}{g(x)}\right] = \lim_{x\to a}\left[\frac{f'(x)}{g'(x)}\right]$$

A limit we hope exists and we hope it is easier to calculate

Note, L'Hôpital Rule also applies to the $\frac{\infty}{\infty}$ form

Examples:

$$\underbrace{\lim_{x\to 0}\left[\frac{sinx}{x}\right]}_{"\frac{0}{0}"} = \lim_{x\to 0}\left[\frac{cosx}{1}\right] = \cos(0) = 1$$

$$\underbrace{\lim_{x\to 0}\left[\frac{e^x - 1}{x^3}\right]}_{"\frac{0}{0}"} = \lim_{x\to 0}\left[\frac{e^x}{3x^2}\right] = \lim_{x\to 0}\left[\frac{e^x}{6x}\right] = \lim_{x\to 0}\left[\frac{e^x}{6}\right] = +\infty$$

$$\underbrace{\lim_{x\to \pi/2}\left[\frac{1 - sinx}{cosx}\right]}_{"\frac{0}{0}"} = \lim_{x\to \pi/2}\left[\frac{-cosx}{-sinx}\right] = \frac{0}{-1} = 0$$

$$\underbrace{\lim_{x\to \infty}\left[\frac{x^2}{e^x}\right]}_{"\frac{\infty}{\infty}"} = \lim_{x\to \infty}\left[\frac{2x}{e^x}\right] = \lim_{x\to \infty}\left[\frac{2}{e^x}\right] = 0$$

Finding other “Indeterminate” Limits

- L’Hôpital Rule applied directly to $\frac{0}{0}$ and $\frac{\pm\infty}{\pm\infty}$
- Also apply to $\infty \cdot 0, \infty - \infty, 1^{\infty}, 0^{0}, \infty^{0}$

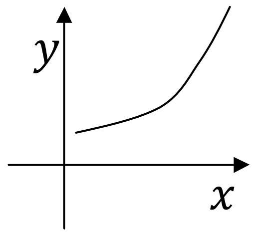
We have to reduce any indeterminate form to either $\frac{0}{0}$ and $\frac{\infty}{\infty}$

Example:

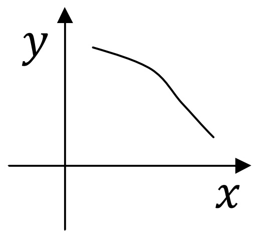
$$\underbrace{\lim_{x\to 0}[x \cdot lnx]}_{"0\cdot(-\infty)"} = \underbrace{\lim_{x\to 0}\left[\frac{lnx}{\frac{1}{x}}\right]}_{"\frac{-\infty}{\infty}"} = \lim_{x\to 0}\left[\frac{\frac{1}{x}}{-x^{-2}}\right] = -\lim_{x\to 0} x = 0$$

Increasing and Decreasing Functions:

Definition (Algebraic): A function f is increasing on same interval I, if for any x_1, x_2 in I $x_1 < x_2$ imply $f(x_1) < f(x_2)$



A function f is decreasing on same interval I, if for any x_1, x_2 in I $x_1 < x_2$ imply $f(x_1) > f(x_2)$



Constant function: not increasing, not decreasing

- f is increasing on an interval ⟺ Graph is rising from left to right
- f is decreasing on an interval ⟺ Graph is falling from left to right

Theorem: If f is continuous on $[a, b]$ and differentiable on (a, b)

Then

$$f'(x) > 0, all\ x \in (a, b) \qquad \Longrightarrow \qquad f\ increasing\ on\ [a, b]$$

$$f'(x) < 0, all\ x \in (a, b) \qquad \Longrightarrow \qquad f\ decreasing\ on\ [a, b]$$

$$f'(x) = 0, all\ x \in (a, b) \qquad \Longrightarrow \qquad f\ constant\ on\ [a, b]$$

Local Maximums and Minimums

- f changes from increasing to decreasing at a relative (or local) maximum point
- f changes from decreasing to increasing at a relative (or local) minimum point

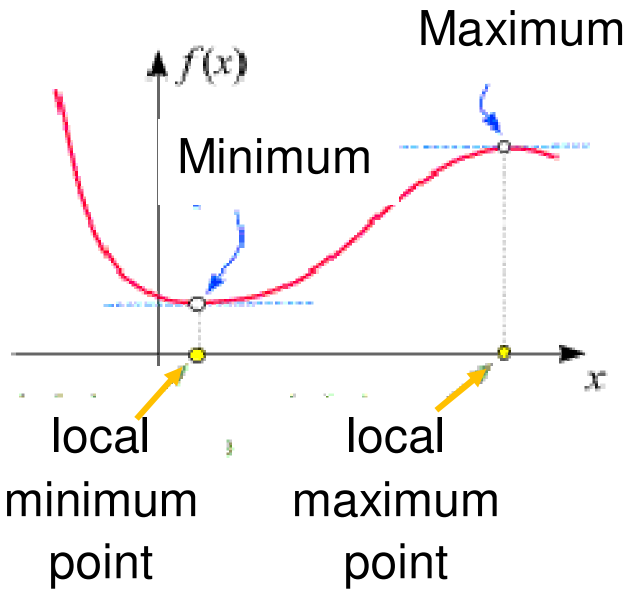
Definition. A function $y = f(x)$ has a local maximum at "c" (some point) (in some interval I) if for all x in I $f(x) \leq f(c)$.

Called a local maximum value for f

Definition. A function $y = f(x)$ has a local minimum at "c" (some point) (in some interval I) if for all x in I $f(x) \geq f(c)$.

Called a local minimum value for f

- Local extremum means either (maximum and minimum)



Term Local (=Relative)

Definition: x_0 in the Domain is a **critical** point for f

If

$$\begin{cases} f'(x_0) = 0 \\ f'(x_0)\ does\ not\ exist \end{cases}$$

Theorem. Let f be defined on I open, containing x_0, f has a local max/min at x_0:

x_0 must be a critical point of f

But !

$$\underbrace{\begin{bmatrix} x_0\ a\ critial\ point \\ of\ f \end{bmatrix}}_{\substack{\boldsymbol{candidates}\ for\ local \\ min/max}} \not\Rightarrow \begin{bmatrix} f\ has\ a\ local \\ extremum\ at\ x_0 \end{bmatrix}$$

Extrema occur at critical points, but not every critical point is an extremum!

To determine the extrema we must do two things:

1. Find the critical points (compute $f'(x)$ and find out where it is either 0 or undefined)

2. "Test" each critical point to determine if it a relative maximum, a relative minimum, or neither

For the second , there are two "tests" available: The first derivative test and The second derivative test.

The 1st derivative Test for local Maximums and Minimums

Observe: [f continuous at critical point x_0]

- Local maximum $f' > 0 \; - \; f' < 0$
- Local minimum $f' < 0 \; - \; f' > 0$

- Using these observations we have the 1st derivative test for local extrema

The 2nd Derivative Test for local Maximums and Minimums

- An alternative to the 1st derivative test. Use only if the 2nd derivative is easy to calculate
- Nice, because instead of looking to the left and right of x_0, you just look directly at x_0

Observe: Assume $f''(x_0)$ exists. [Thus, $f'(x_0)$ must exist]

So,

$$\begin{bmatrix} f'(x_0) = 0 \\ and\ f''(x_0) > 0 \end{bmatrix} \Longrightarrow \begin{bmatrix} f\ has\ a\ local \\ minimum\ at\ x_0 \end{bmatrix}$$

$$\begin{bmatrix} f'(x_0) = 0 \\ and\ f''(x_0) < 0 \end{bmatrix} \Longrightarrow \begin{bmatrix} f\ has\ a\ local \\ maximum\ at\ x_0 \end{bmatrix}$$

$$\begin{bmatrix} f'(x_0) = 0 \\ and\ f''(x_0) = 0 \end{bmatrix} \Longrightarrow [Inconclusive]$$

Example: Find all local extrema of the function:

$$f(x) = -2x^3 + 3x^2 + 12x + 10$$

Solution:

$$f'(x) = -6x^2 + 6x + 12$$

$$-6x^2 + 6x + 12 = 0$$

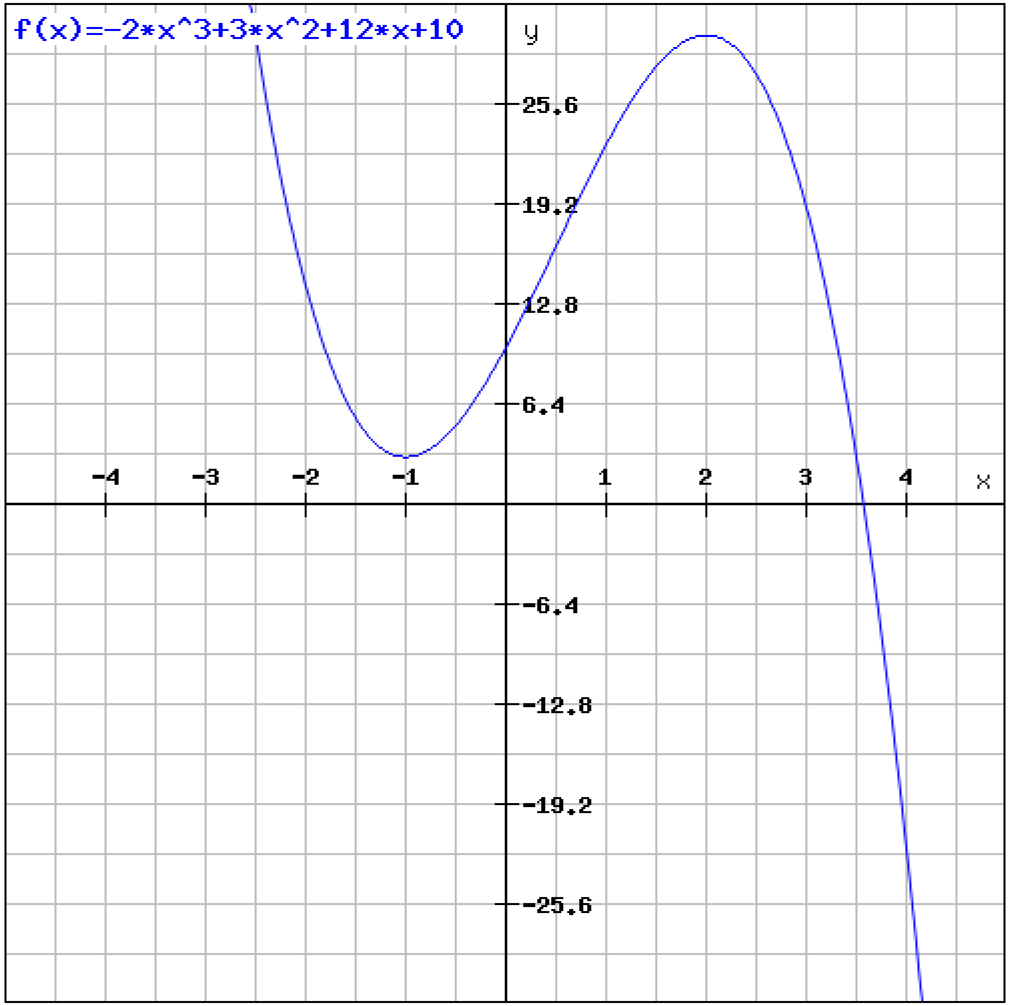
$$x_{1,2} = \frac{-6 \pm 18}{-12}$$

$$x_1 = 2, x_2 = -1$$

$$f''(x) = -12x + 6$$

$$x_1 = 2: \quad f''(x) = -12 \cdot 2 + 6 = -18 < 0: local\ maximum$$

$$x_2 = -1: \quad f''(x) = -12 \cdot (-1) + 6 = 18 > 0: local\ minimum$$



f(x)=-2*x^3+3*x^2+12*x+10
y
25.6
19.2
12.8
6.4
-4
-3
-2
-1
1
2
3
4
x
-6.4
-12.8
-19.2
-25.6

Global (Absolute) Maximums and Minimums

Consider: the function $f(x)$, I is same Interval in the Domain of f and $x_0 \in I$

Definition:

- f has a global maximum at x_0 if $f(x_0) \geq f(x)$ at $x \in I$
- f has a global minimum at x_0 if $f(x_0) \leq f(x)$ at $x \in I$

We say "global extremum" for either

Global extrema on (finite) closed Intervals

Extreme Value Theorem

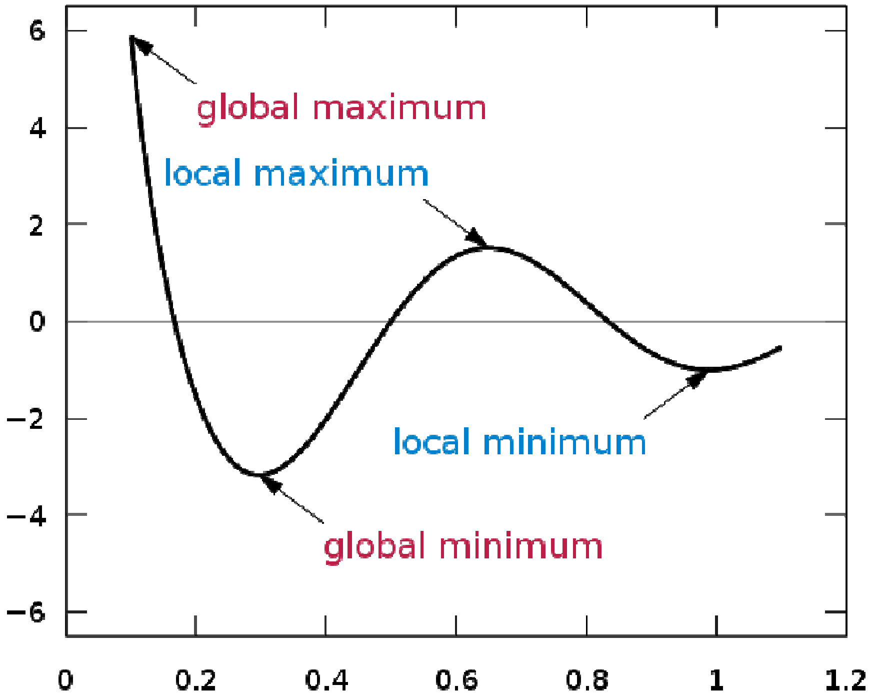
If f is continuous on close I $[a, b]$ [both hypothesis necessary], Then f has both a global maximum and global minimum [guaranteed!] – "Existence Theorem"

Further Theorem: Suppose f has a global extremum on an Interval (a, b) open. Then that extremum must occur at a critical point.

Summary:

$$\begin{bmatrix} f\ continuous \\ on\ [a,b] \end{bmatrix} \Rightarrow \begin{cases} f\ has\ both\ global\ extrema \\ This\ occur\ at \\ either\ a,b\ [endpoint] \\ or\ when\ f'(x)=0 \\ or\ f'does\ not\ exist \end{cases}$$

1. Find all the critical points of f $[a, b]$
2. Evaluate f as these points, and at a and b
3. Largest value=global maximum
   Smallest value=global minimum



http://en.wikipedia.org/wiki/File:Extrema_example_original.svg

Optimization Problem

Applied Maximum and Minimum problems

Called "optimization" (find the best")

A strategy

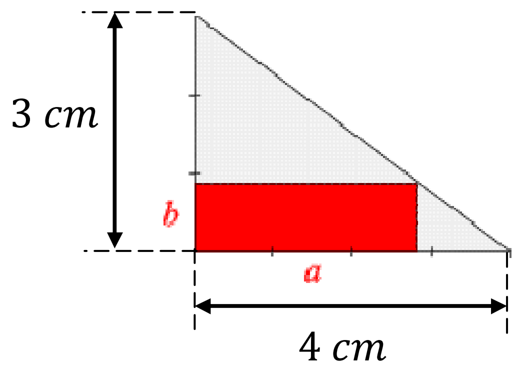
- Draw a sketch +label relevant quantities
- Find a formula for the one quantity to be maximized or minimized
- Use given information to write that formula as a function of one variable
- Find the domain of that variable
- Use the derivative to find the desired global max/min

Example: What is the biggest Rectangle you can put inside a given triangle?

Given a right triangle of altitude $3\ cm$ an base $4cm$

Find a dimension of the rectangle of maximum area that can be inscribed in this triangle with one side along the base.

- A sketch

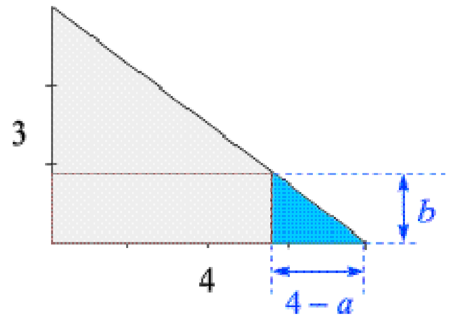


- A formula to be maximized

$$f = a \cdot b$$

We seek the maximum to the product $a \cdot b$ .We need to find a so that f is maximized

- The formula as a function of one variable



$$\frac{3}{4} = \frac{b}{4-a}$$

$$b = \frac{3(4-a)}{4}$$

$$f = a \cdot b = \frac{3a(4-a)}{4}$$

- Domain of a: $0 < a < 4$

- The derivative used

$$f'(a) = 3 - 1{,}5a = 0$$

$$a = 2$$

maximum or minimum?

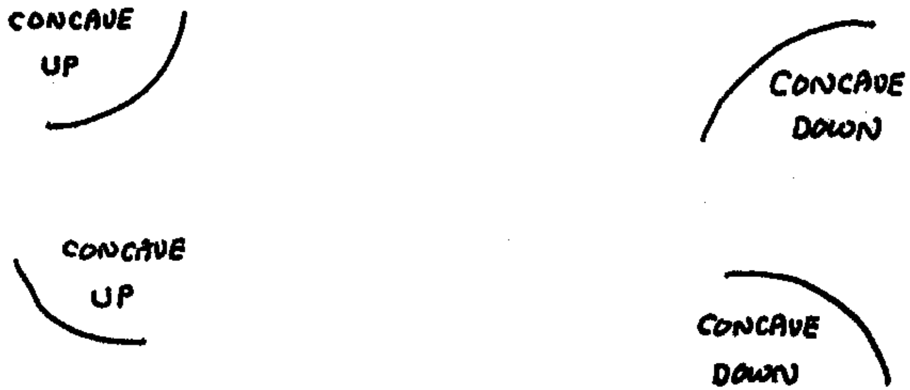
$f'' = -1{,}5 < 0$ - maximum

-

$$b = \frac{3(4-a)}{4} = 1{,}5$$

Function Concave Up and Concave Down:

- f can increase (or decrease) in two different way: concave up and concave down



- a point at which f changes from concave up to concave down or from concave down to concave up is called an **inflection** point.

Function Concave Up and Concave Down: The 2nd derivative applied

Definition: Let f have a derivative on open interval I

- f concave up on I means f' is increasing on I
- f concave down on I means f' is decreasing on I

To tell if a function (later f') is increasing/decreasing, we check its first derivative of (f'):

$$(f')' = f''$$

Theorem:

Suppose f is twice differentiate on I

$$\begin{cases} f''(x) > 0 \\ all\ x \in I \end{cases} \Rightarrow f\ is\ concave\ up\ in\ I$$

$$\begin{cases} f''(x) < 0 \\ all\ x \in I \end{cases} \Rightarrow f\ is\ concave\ down\ in\ I$$

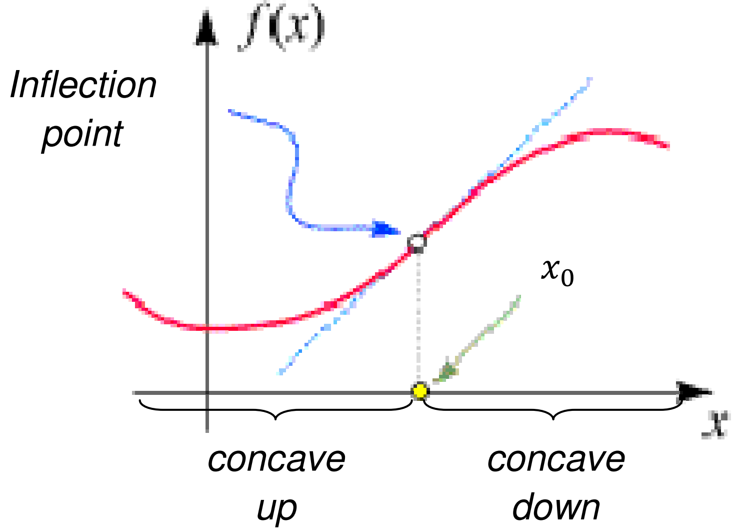
When Concavity Changes: Inflection Points

Definition:

If f is continuous on open I and concavity changes at $(x_0, f(x_0))$

Then we say: f has an inflection point at x_0 and $(x_0, f(x_0))$ is that inflection point

$f''(x_0) = 0$ gives candidates for inflection points, but no guaranties

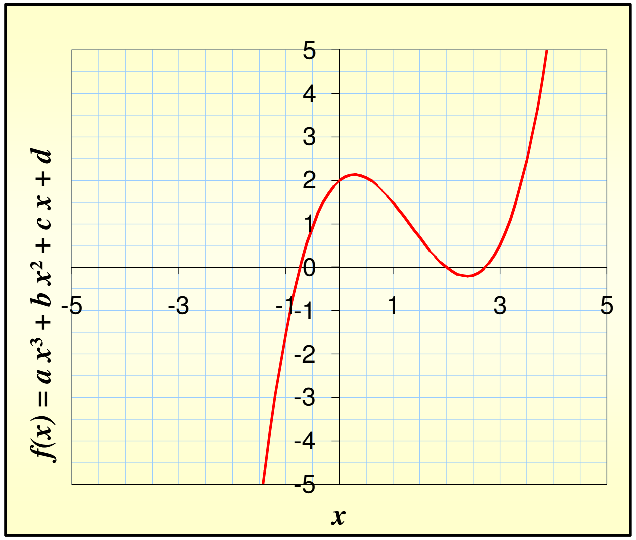


Examples:

function	**1.derivative**	**2. derivative**	**Concave up/down?**
$f(x) = x^2$	$2x$	$2 > 0$	concave up
$f(x) = -x^2$	$-2x$	$-2 < 0$	concave down
$f(x) = (e^{2x} + 4e^{-x})^2$	$4e^{4x} + 8e^{x} - 32e^{-2x}$	$16e^{4x} + 8e^{x} + 64e^{-2x} > 0$	concave up

Example:

$$f(x) = 0{,}5x^3 - 2x^2 + x + 2$$



$$f'(x) = 1{,}5x^2 - 4x + 1;\ f''(x) = 3x - 4;\ f'''(x) = 3 > 0$$

$$3x - 4 < 0:\ x < \frac{4}{3}; \qquad \left]-\infty; \frac{4}{3}\right] : concave\ up$$

$$3x - 4 > 0:\ x > \frac{4}{3}; \qquad \left[\frac{4}{3};\ +\infty\right[: concave\ down$$

$$x = \frac{4}{3} : inflection\ point$$

What to look for in a graph:

With Algebra:

- Domain and Range
- x intercepts
- y intercepts
- symmetrie

With Limits:

- Asymptotes
- End Behavior $x \to -\infty, x \to \infty$

With derivatives:

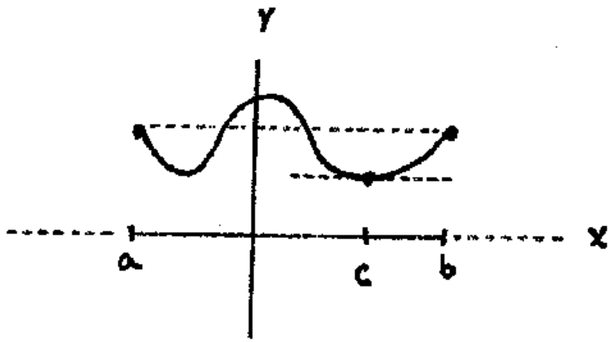
- Increasing/decreasing
- Local Extrema
- Concave up/down
- Inflection Points

The mean Value Theorem for Derivatives

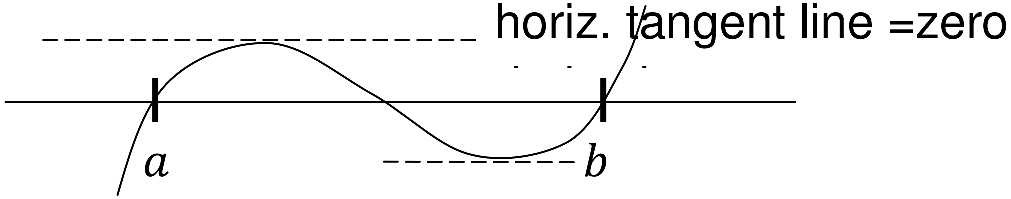
A special Case of the Mean Value: Rolle's Theorem

If f is continuous on $[a, b]$, f is differentiable on (a, b), and $f(a) = f(b)$

Then there is at least one c in (a, b) such that $f'(c) = 0 \leftarrow \begin{bmatrix} slope\ of \\ secant\ line\ between \\ (a, f(a))and\ (b, f(b)) \end{bmatrix}$



Proof for $f(a) = 0 = f(b)$



(1) Suppose $f(x) = 0$ for all x in (a, b) [a constant function]. Then $f'(c) = 0$ for all c in (a, b)

(2) Suppose $f(x) > 0$ for some point in (a, b). Since f is continuous on $[a, b]$

[Extreme Value Theorem] $\Rightarrow f$ has a global max on $[a, b]$ in fact on (a, b)[because $f(a) = 0$ and $f(b) = 0$]

Since f is differentiable on (a, b), there must be a critical point c in (a, b), where $f'(c) = 0$

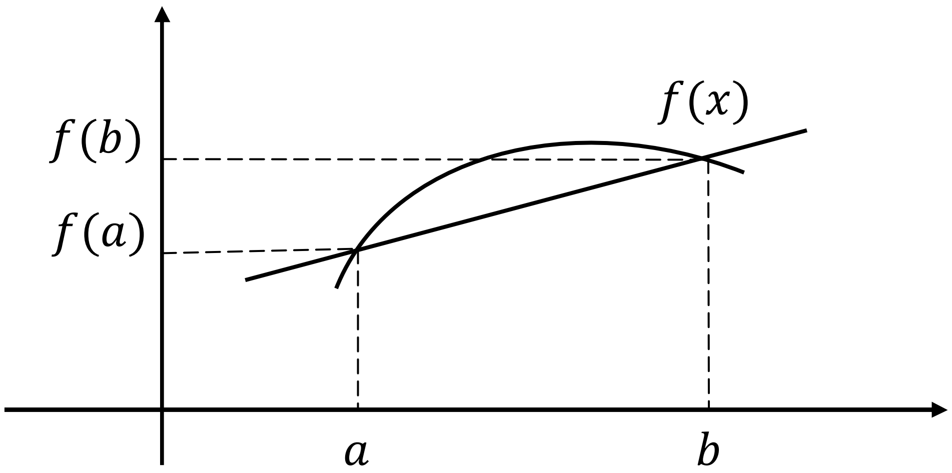
(3) (The $f(x) < 0$, is similar)

The Full Mean Value Theorem of Derivatives

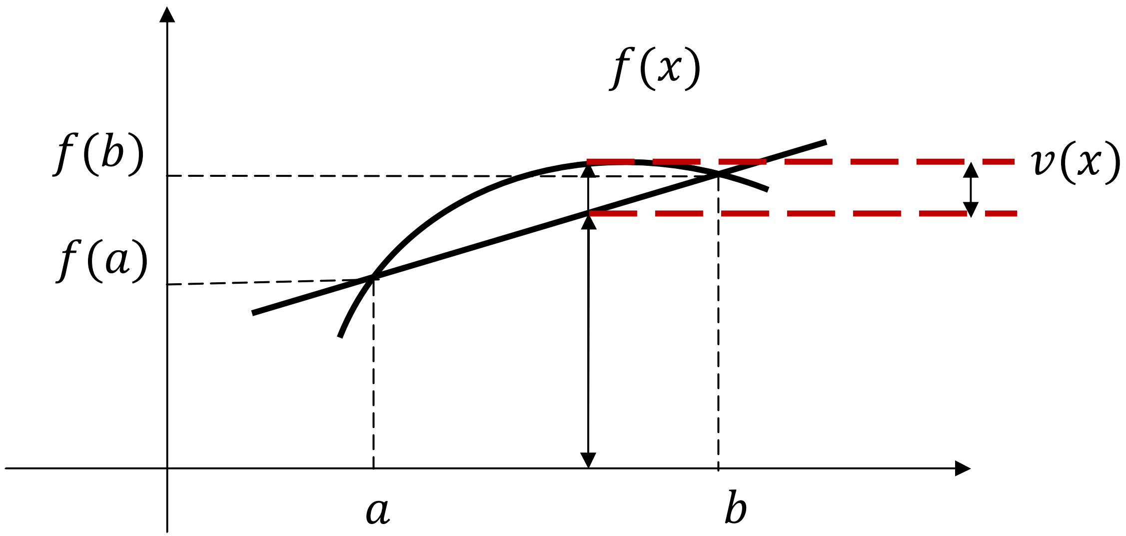
If f is continuous on $[a, b]$ f is differentiable on (a, b)

Then there is least one point c in (a, b) at which the tangent line is parallel to the secant line joining the points $(a, f(a))$ and $(b, f(b))$, i.e. at which

$$\underbrace{f'(c)}_{\substack{tangent\ slope \\ at\ c}} = \underbrace{\frac{f(b) - f(a)}{b - a}}_{\substack{secant\ slope \\ between\ (a,f(a)) \\ and\ (b,f(b))}}$$



Proof of MVT for Derivatives



Secant lines

$$y - f(a) = \left[\frac{f(b) - f(a)}{b - a}\right](x - a); \;\; y = \left[\frac{f(b) - f(a)}{b - a}\right](x - a) + f(a)$$

Let v be a function: v=[height of f]-[hight of secant line]

$$v(x) = \underbrace{f(x) - \left[\frac{f(b) - f(a)}{b - a}(x - a) + f(a)\right]}_{Difference\ of\ two\ heghts}$$

Since f is continuous on $[a, b]$ so is $v(x)$ [because a secant is just a line – continuous]

Observe $v(a) = 0$ and $v(b) = 0$

So v satisfied Rolle's Theorem, meaning there is c in (a, b) with $v'(c) = 0$

But

$$v'(x) = f'(x) - \left[\frac{f(b) - f(a)}{b - a}\right]$$

$$0 = v'(c) = f'(c) - \left[\frac{f(b) - f(a)}{b - a}\right]$$

So

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

Direct Consequences of the Mean value Theorem

(1) Consequence: Theorem:

(recall – previously not proven)

Suppose f is continuous at $[a, b]$, f differentiable on (a, b)

a) $\begin{bmatrix} f'(x) > 0 \\ all\ x\ in\ (a, b) \end{bmatrix} \Longrightarrow f\ increase\ on\ [a, b]$

b) $\begin{bmatrix} f'(x) < 0 \\ all\ x\ in\ (a, b) \end{bmatrix} \Longrightarrow f\ decrease\ on\ [a, b]$

c) $\begin{bmatrix} f'(x) = 0 \\ all\ x\ in\ (a, b) \end{bmatrix} \Longrightarrow f\ constant\ on\ [a, b]$

Proof of Part (a) only

Let x_1, x_2 be in $[a, b]$ with $x_1 < x_2$ so $[x_2 - x_1 > 0]$

We must show $f(x_1) < f(x_2)$

Since the MVT hypothesis holds on$[a, b]$. The Theorem also holds on $[x_1, x_2]$

So there is a c in (x_1, x_2) such that

$$f'(c) = \left[\frac{f(x_2) - f(x_1)}{x_2 - x_1}\right]$$

$$f(x_2) - f(x_1) = \underbrace{\underbrace{f'(c)}_{positive}\ \underbrace{(x_2 - x_1)}_{positive}}_{positive} > 0$$

$$f(x_2) > f(x_1)$$

(2) Consequence: Constant Difference Theorem

If f, g are differentiable on Interval I and $f'(x) = g'(x)$ for all x in I,

Then for all x in I

$$f(x) - g(x) = k\ (constant)$$

meaning

$$f(x) = g(x) + k$$

Two function with the same derivative differ at most by a constant in I say, $x_1 < x_2$

Proof:

Let x_1, x_2 be different in I, say $x_1 < x_2$

Since f, g are differentiable in I, then f, g continuous in I

So, f, g are differentiable on (x_1, x_2) and continuous on $[x_1, x_2]$

The same hold true for

$$F(x) = f(x) - g(x)$$

Now,

Our hypothesis

$$F'(x) = f'(x) - g'(x) = 0$$

By the previous Consequence: Theorem (1c) we know $F(x) = k$ constant

So, $f(x) - g(x) = k$ at both x_1 and x_2

Since x_1, x_2 arbitrary in I $f(x) = g(x) = k$ for all x in I.

A function of two variables

A function of two variables x and y is a rule which assigns to each ordered pair (x, y) of real numbers in some subset of the xy-plane (called the domain of the function) exactly one real number

$$z = f(x, y)$$

called the value of f at (x, y).

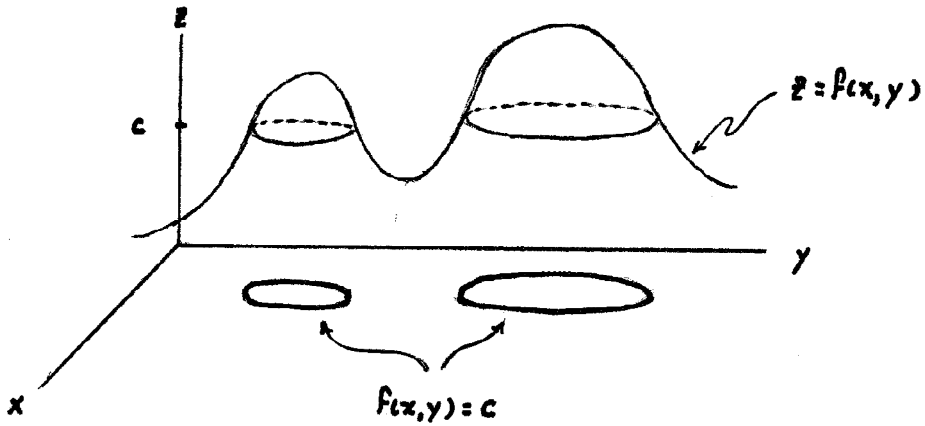
The value of f depends on two different parameters

Example: The temperature at the certain point on the surface of the earth $f(x, y)$, where x and y are longitude and latitude.

The graph of f

The graph of f is a surface in space. So for each value of x and y we have x, y in the (x, y) −plane, then we'll plot the point in space at position x, y: $z = f(x, y)$

It is possible to obtain something like a "picture" of a function $z = f(x, y)$ without drawing its graph in space. It is the **contour** plot. The graph is sliced by horizontal planes. It is a representing the function of two variables by the map.

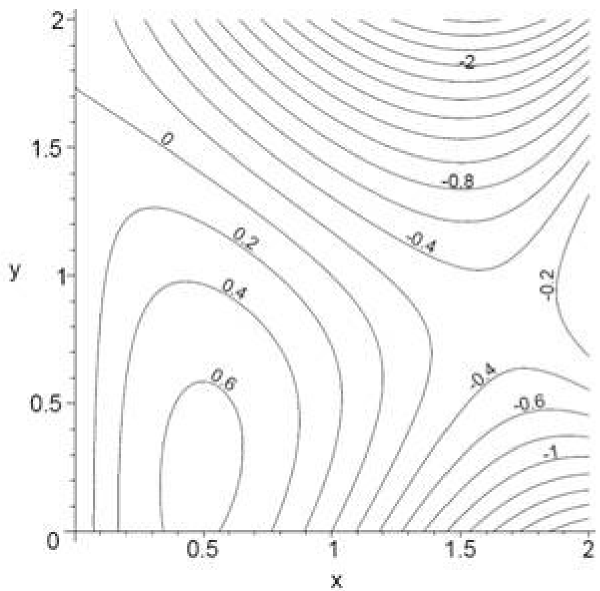


There are a bunch of curves. A **level** curve for $z = f(x, y)$ is a curve in the x, y -plane on which the function takes only one value, i.e. with an equation of the form

$$f(x, y) = c$$

for constant c

Draw enough of these, label each with the c it came from (so that you know how height it should be lifted to get to the graph) and you have some idea what the surface looks like.



Limits and continuity for function of two variables.

Recall:

$$\lim_{x \to x_0} f(x) = L$$

If $f(x)$ can be made as close as we like to L by choosing x sufficiently close (but not equal) to x_0

$\lim_{x \to x_0} f(x) = L$ exists if and only if both

$$\lim_{x \to x_0^-} f(x) = L$$

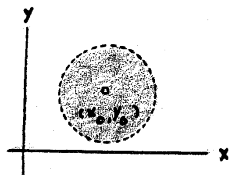
and

$$\lim_{x \to x_0^+} f(x) = L$$

are equal

For $f(x, y)$ the definition looks essentially the same:

Given $f(x, y)$ an a point (x_0, y_0) in the plane with f defined at least "near" (x_0, y_0)

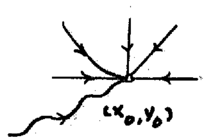


We say that

$$\lim_{(x,y)\to(x_{0,}y_0)} f(x, y) = L$$

if $f(x, y)$ can be made as close as we like to L choosing (x, y) sufficiently close (but not equal) to (x_0, y_0).

This time, however, instead of just two there are infinitely many "approaches" to $\left(x_{0,}, y_{0,}\right)$ and, in order for the limit to exist, they must all give the same result.



Continuity

Recall: $f(x)$ is continuous at x_0 if $\lim_{x \to x_0} f(x) = f(x_0)$

Implicit in this is

- x_0 is in the domain of $f(x)$ so $f(x_0)$ exists
- $\lim_{x \to x_0} f(x)$ exists
- these two are the same

For function of two variables the definition is the same

$f(x, y)$ is continuous at (x_0, y_0) if

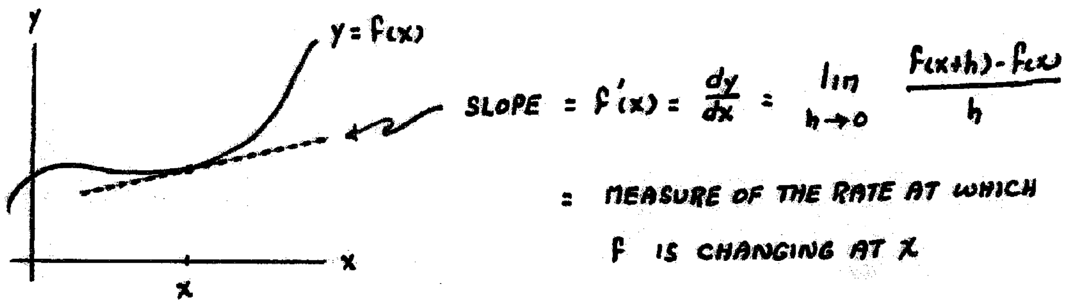
$$\lim_{(x,y) \to (x_{0,} y_0)} f(x, y) = f(x_{0,} y_0)$$

If this is true for every $(x_{0,} y_0)$ in the domain of $f(x, y)$ we say simply that $f(x, y)$ is continuous

- Polynomials are continuous everywhere
- Rational functions are continuous wherever the denominator is nonzero
- Sums, differences and products of continuous functions are continuous
- Quotients of continuous functions are continuous wherever the denominator is nonzero
- If $f(x, y)$ is continuous and $g(u)$ is a continuous function of one variable, then $g(f(x, y))$ is continuous

Partial Derivatives

Recall: given $y = f(x)$ and x in its domain



Now suppose $y = f(x, y)$ and (x, y) is a point in its domain.

"Rate at which f is changed at (x, y)" makes no sense since f can change at different rate in different directions at (x, y)

Partial Derivatives: Rates of changes in the x-direction and in the y-direction

Slope of a tangent line in x-direction = **partial derivative** of f with respect to x

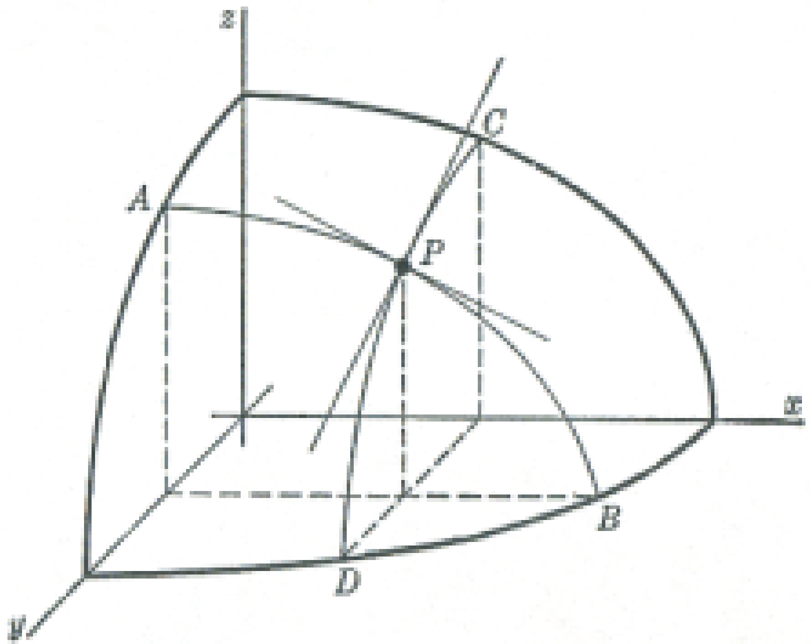
$$= \frac{\partial f}{\partial x} = \lim_{h \to 0} \frac{f(x+h, y) - f(x, y)}{h}$$

- hold y fixed and differentiate with respect to x as usual.

Slope of a tangent line in y-direction = partial derivative of f with respect to y

$$= \frac{\partial f}{\partial y} = \lim_{h \to 0} \frac{f(x, y+h) - f(x, y)}{h}$$

- hold y fixed and differentiate with respect to y with usual.


z
C
A
P
x
B
D
y

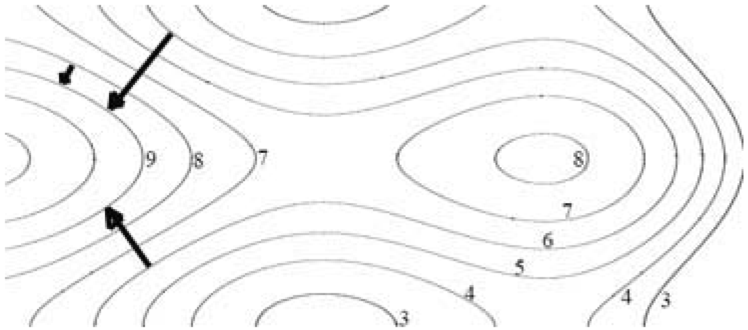
Examples:

$$f(x,y) = x \cdot siny, \qquad \frac{\partial f}{\partial x} = siny, \qquad \frac{\partial f}{\partial y} = x \cdot cosy$$

$$f(x,y) = x^2 + y^2, \qquad \frac{\partial f}{\partial x} = 2x, \qquad \frac{\partial f}{\partial y} = 2y$$

Gradient

The **gradient** of a function f points in the direction of the greatest rate of increase of the function, and whose magnitude is that rate of increase.



The **gradient** of f:

$$\nabla f = grad\ f = \begin{pmatrix} \dfrac{\partial f}{\partial x} \\ \dfrac{\partial f}{\partial y} \end{pmatrix}$$

The **gradient** of f at the point (x_0, y_0):

$$\nabla f(x_0, y_0) = \begin{pmatrix} \dfrac{\partial f}{\partial x}(x_0, y_0) \\ \dfrac{\partial f}{\partial y}(x_0, y_0) \end{pmatrix}$$

Tangent plane

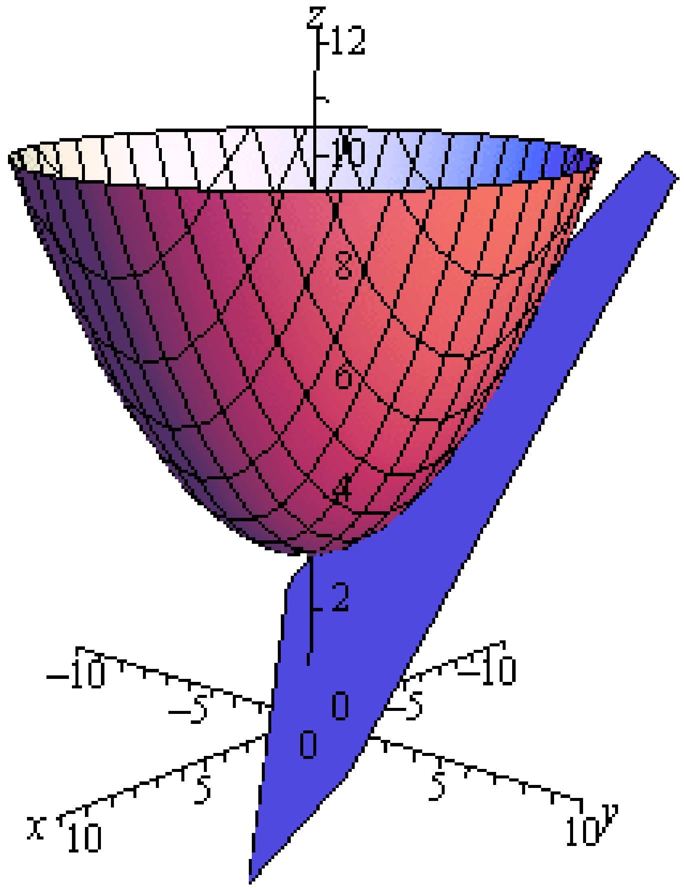
Let (x_0, y_0)be any point of a surface function $z = f(x, y)$ Then the surface has a nonvertical tangent plane at (x_0, y_0) with equation

$$T_{(x_0,y_0)} = f(x_0, y_0) + \begin{pmatrix} \frac{\partial f}{\partial x}(x_0, y_0) \\ \frac{\partial f}{\partial y}(x_0, y_0) \end{pmatrix} \cdot \begin{pmatrix} x - x_0 \\ y - y_0 \end{pmatrix} = f(x_0, y_0) + \underbrace{\nabla f(x_0, y_0)}_{\substack{Gradient\ at\ point \\ (x_0,y_0)}} \begin{pmatrix} x - x_0 \\ y - y_0 \end{pmatrix}$$

A tangent plane to a function $f(x_0, y_0)$ at the point (x_0, y_0) is a plane that just touches the graph of the function at the point $\big((x_0, y_0), f(x_0, y_0)\big)$.

Approximation formula = the graph is close to its tangent plane.

Tangent plane



http://tutorial.math.lamar.edu/Classes/CalcIII/TangentPlanes.aspx

Example: Find the equation of a tangent plane to:

$$f(x, y) = x^2 + y^2$$

At the point $(x_0, y_0) = (1,2)$

Solution:

$$\nabla f(x_0, y_0) = (2x \quad 2y)(1,2) = \begin{pmatrix} 2 \\ 4 \end{pmatrix}$$

$$T(x, y) = f(1,2) + \nabla f(1,2) \begin{pmatrix} x - 1 \\ y - 2 \end{pmatrix} = 5 + (2 \quad 4) \begin{pmatrix} x - 1 \\ y - 2 \end{pmatrix} = 5 + 2(x - 1) + 4(y - 2) =$$
$$= -5 + 2x + 4y$$

The total differential

The total differential of the function of two variables df

$$df = \frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial y} dy$$

The total differential gives the full information about rates of change of the function in the x-direction and in the y-direction.

Some alternate notation:

$$\frac{\partial f}{\partial x} = \frac{\partial z}{\partial x} = \frac{\partial}{\partial x} f(x, y) = f_x = D_x f = D_1 f = \cdots$$

$$\frac{\partial f}{\partial y} = \frac{\partial z}{\partial y} = \frac{\partial}{\partial y} f(x, y) = f_y = D_y f = D_2 f = \cdots$$

Second order derivatives: $\boldsymbol{f(x, y)}$

$$\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2 f}{\partial x^2} = f_{xx}$$

$$\frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial^2 f}{\partial y^2} = f_{yy}$$

$$\left.\begin{aligned} \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) &= \frac{\partial^2 f}{\partial y \partial x} = f_{yx} \\ \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) &= \frac{\partial^2 f}{\partial x \partial y} = f_{xy} \end{aligned}\right\} \begin{gathered} mixed\ second \\ order \\ partial \\ derivatives \end{gathered}$$

Examples:

$$f(x, y) = x^3 y - x^2 y^2$$

$$f_x = 3x^2 y - 2xy^2,\ \ f_y = x^3 - 2x^2 y$$

$$f_{xx} = 6xy - 2y^2,\ \ f_{yy} = -2x^2$$

$$f_{xy} = 3x^2 - 4xy,\ \ f_{yx} = 3x^2 - 4xy$$

Local maxima and minima

At a local max or min, $f_x = 0$ and $f_y = 0$

Definition of a critical point: (x_0, y_0) where $f_x = 0$ and $f_y = 0$

A critical point may be a local minimum, local maximum, or saddle.

Second derivative test

Goal: determine type of a critical point, and find the local min/max.
Note: local min/max occur at a critical points

General case: second derivative test.
We look at second derivatives:

$$f_{xx} = \frac{\partial^2 f}{\partial x^2};\ f_{xy} = \frac{\partial^2 f}{\partial x \partial y} = f_{yx} = \frac{\partial^2 f}{\partial y \partial x};\ f_{yy} = \frac{\partial^2 f}{\partial y^2}$$

The **Hessian matrix** (or simply the **Hessian**) is the square matrix of second-order partial derivatives of a function

$$H(f) = \begin{pmatrix} \frac{\partial^2 f}{\partial x^2} & \frac{\partial^2 f}{\partial x \partial y} \\ \frac{\partial^2 f}{\partial x \partial y} & \frac{\partial^2 f}{\partial y^2} \end{pmatrix} = \begin{pmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{pmatrix}$$

Given is f and a critical point (x_0, y_0),
then:
if

$$f_{xx}(x_0, y_0) \cdot f_{yy}(x_0, y_0) - f_{xy}(x_0, y_0) \cdot f_{yx}(x_0, y_0) > 0$$

then

- if

$$f_{xx}(x_0, y_0) > 0$$

  local min

- if

$$f_{xx}(x_0, y_0) < 0$$

local max.
if

$$f_{xx}(x_0, y_0) \cdot f_{yy}(x_0, y_0) - f_{xy}(x_0, y_0) \cdot f_{yx}(x_0, y_0) < 0$$

then saddle
if

$$f_{xx}(x_0, y_0) \cdot f_{yy}(x_0, y_0) - f_{xy}(x_0, y_0) \cdot f_{yx}(x_0, y_0) = 0$$

then can’t conclude

Example:

$$f(x,y) = y^3 + x^2(y+1) - 12y + 11$$

$$f_x = (y+1)2x \qquad f_y = 3y^2 + x^2 - 12$$

$$f_{xx} = 2y + 2 \quad f_{yy} = 6y$$

$$f_{yx} = f_{xy} = 2x$$

Critical points candidates:

$$f_x = (y+1)2x = 0 \quad f_y = 3y^2 + x^2 - 12 = 0$$

$$(x_1, y_1) = (3, -1)\ ;\ (x_2, y_2) = (-3, -1);\ (x_3, y_3) = (0, -2)\ ;\ (x_4, y_4) = (0,2)$$

$$(x_1, y_1) = (3, -1): \frac{\partial^2 f}{\partial x^2}\frac{\partial^2 f}{\partial y^2} - \left(\frac{\partial^2 f}{\partial x \partial y}\right)^2 = 0 - 36 = -36 < 0 \; saddle$$

$$(x_2, y_2) = (-3, -1): \frac{\partial^2 f}{\partial x^2}\frac{\partial^2 f}{\partial y^2} - \left(\frac{\partial^2 f}{\partial x \partial y}\right)^2 = 0 - 36 = -36 < 0 \; saddle$$

$$(x_3, y_3) = (0, -2): \frac{\partial^2 f}{\partial x^2}\frac{\partial^2 f}{\partial y^2} - \left(\frac{\partial^2 f}{\partial x \partial y}\right)^2 = 24 - 0 = 24 > 0; \; \frac{\partial^2 f}{\partial x^2} = -2 < 0 \; maximum$$

$$(x_4, y_4) = (0, 2): \frac{\partial^2 f}{\partial x^2}\frac{\partial^2 f}{\partial y^2} - \left(\frac{\partial^2 f}{\partial x \partial y}\right)^2 = 72 - 0 = 72 > 0; \; \frac{\partial^2 f}{\partial x^2} = 6 > 0 \; minimum$$