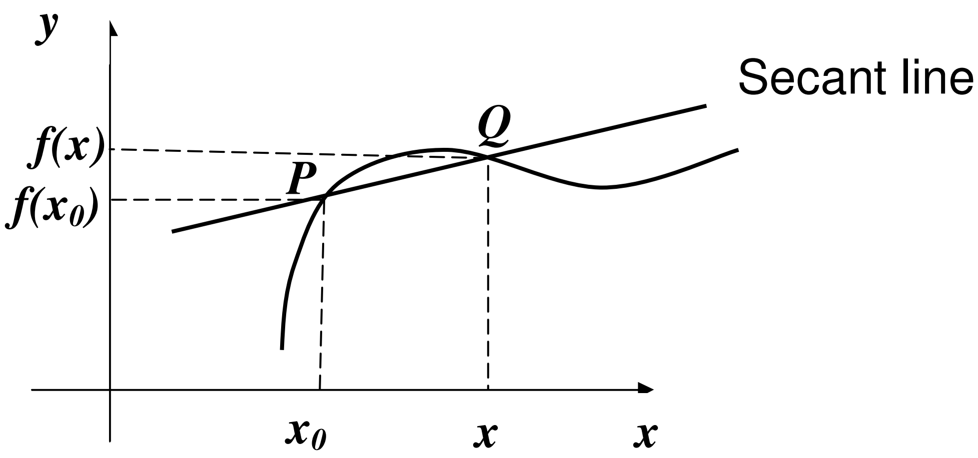
The Derivative of a Function

Measuring Rates of Change of a function $f(x)$

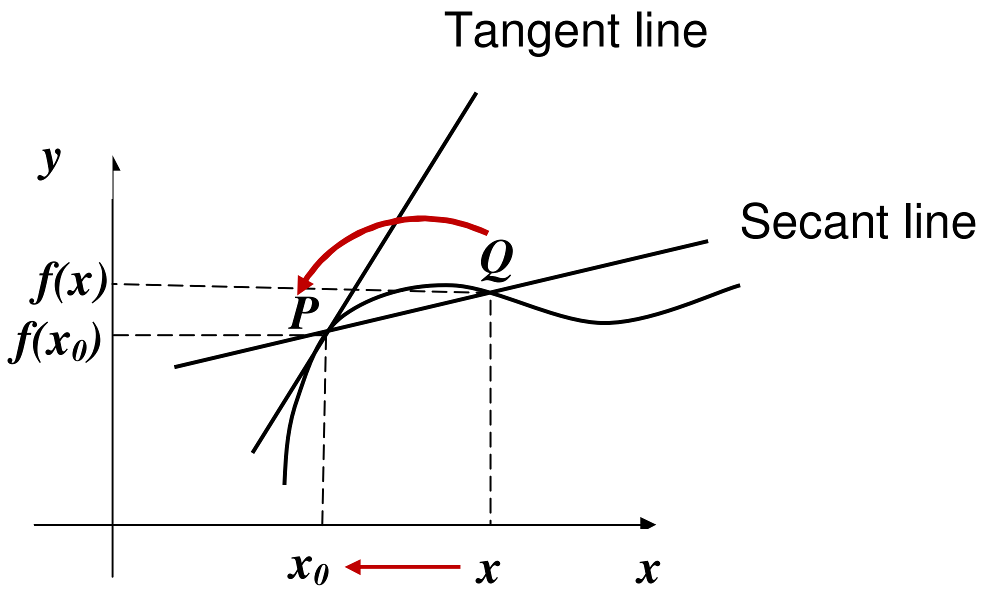


Average rate of change of y with respect to x over $[x_0, x]$

$$= r_{average} = \frac{\text{"}change\ in\ y\text{"}}{\text{"}change\ in\ x\text{"}} = \frac{f(x) - f(x_0)}{x - x_0}$$

- Slope of secant line through $x_0, f(x_0)$ and $x, f(x)$

Instantaneous rate of change of y with respect to x at point x_0

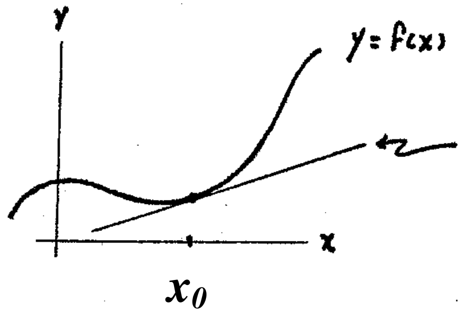


l

$$r_{instantaneous} = \lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0}$$

- Slope of tangent line at $x_0, f(x_0)$ [provided the limit exists]

Slope of Tangent Lines



Definition:

$$[Tangent\ Slope\ at\ x_0] = \lim_{x \to x_0} [Secant\ slope\ between\ x_0\ and\ x]$$

So,

$$m_{Tangent} = \lim_{x \to x_0} \left[\frac{f(x) - f(x_0)}{x - x_0} \right]$$

[provided the limit exists]

Since the tangent line passes through $(x_0, f(x_0))$, its equation is

$$y - f(x_0) = m_{tangent}(x - x_0)$$

Alternate notation:

$$x = x_0 + h; \quad m_{Tangent} = \lim_{h \to 0} \left[\frac{f(x_0 + h) - f(x_0)}{h} \right]$$

What is a Derivative

Definition: The function f' [f prime of x] derived from f and defined by

$$f'(x) = \lim_{h \to 0}\left[\frac{f(x+h) - f(x)}{h}\right]$$

is called the derivative of f with respect to (wrt) x

Exercise:

Find $f'(x)$, if $f(x) = x^2 + 1$

Solution:

$$f'(x) = \lim_{h \to 0} \left[\frac{f(x+h) - f(x)}{h} \right] = \lim_{h \to 0} \left[\frac{(x+h)^2 + 1 - (x^2+1)}{h} \right] =$$

$$\lim_{h \to 0} \left[\frac{1}{h} (x^2 + 2hx + h^2 + 1 - x^2 - 1) \right] = \lim_{h \to 0} \left[\frac{2hx + h^2}{h} \right] = 2x$$

Exercise:

Let check that the tangent slope of $f(x) = mx + b$ is "m" everywhere

Solution:

$$f'(x) = \lim_{h \to 0} \left[\frac{f(x+h) - f(x)}{h} \right] = \lim_{h \to 0} \left[\frac{1}{h} \big(m(x+h) + b - (mx + b) \big) \right] =$$

$$= \lim_{h \to 0} \left[\frac{1}{h} mh \right] = \lim_{h \to 0} m = m$$

Functions: Differentiable (or not!) at a single point?

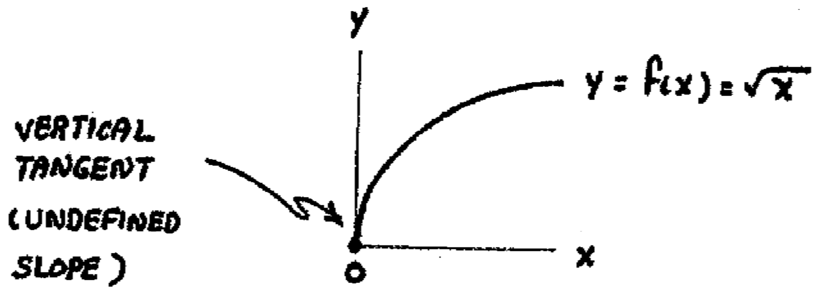
We say: f is differentiable at x_0 [has a derivative at x_0] if $f'(x_0)$ exists.

The process of finding derivatives of function is called <u>differentiation</u>

If a function has a derivative at a point it is said to be <u>differentiable</u> at that point

e.g. $f(x) = \sqrt{x}$ is differentiable at every point in its domain except $x = 0$

Geometric reason:



A function differentiable at a point is continuous at that point

Theorem: If f is differentiable at x_0 then f is continuous at x_0

Proof: Since f is differentiable at x_0 we know

$$f'(x_0) = \lim_{h\to 0}\left[\frac{f(x_0+h) - f(x_0)}{h}\right]$$

exists

To show f is continuous at x_0 we must show [definition of a continuous function]

$$\lim_{x\to x_0} f(x) = f(x_0)$$

We can rewrite:

$$\lim_{x\to x_0}[f(x) - f(x_0)] = 0$$

Rewriting once more, we need to show with $x = x_0 + h$

$$\lim_{h \to 0} [f(x_0 + h) - f(x_0)] = 0$$

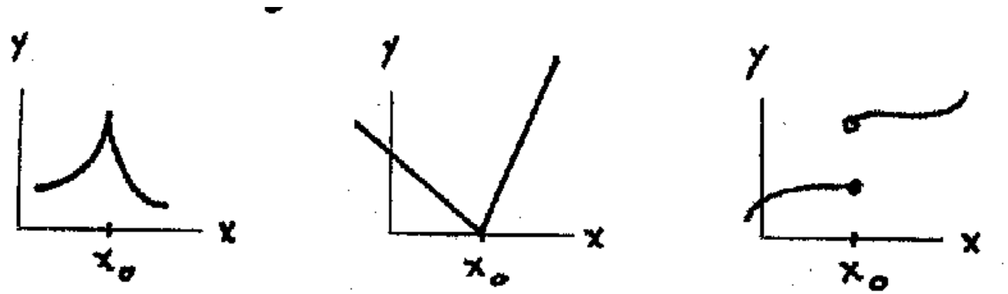
$$\lim_{h \to 0} [f(x_0 + h) - f(x_0)] = \lim_{h \to 0} \left[(f(x_0 + h) - f(x_0)) \cdot \underbrace{\frac{h}{h}}_{=1} \right] = \lim_{h \to 0} \left[\underbrace{\frac{f(x_0 + h) - f(x_0)}{h}}_{f'(x_0)} \cdot h \right]$$

$$= f'(x_0) \cdot 0 = 0$$

So, if $f(x)$ is not continuous at x_0, then $f(x)$ is not differentiable at x_0

f can fail to be differentiable!

Here are the ways in which $f(x)$ can fail to be differentiable at x_0



Example: $f(x) = |x|$ is not differentiable at $x = 0$

$$\lim_{h \to 0} \frac{f(0+h) - f(0)}{h} = \lim_{h \to 0} \frac{|h|}{h}$$

which does not exist because

$$\frac{|h|}{h} = \begin{cases} 1, h > 0 \\ -1, h < 0 \end{cases}$$

Functions Differentiable on an Interval

- On open intervals: a function must be differentiable at each point (2-sided limit)
- On interval with endpoints: a function must be differentiable at each point on the open interval (2-sided limit) and have a left/right hand limits at the end points

Definition:

Left Hand Derivative

$$f_{-}'(x) = \lim_{h \to 0^{-}} \left[\frac{f(x+h) - f(x)}{h} \right]$$

Right Hand Derivative

$$f_{+}'(x) = \lim_{h \to 0^{+}} \left[\frac{f(x+h) - f(x)}{h} \right]$$

Other Derivative Notations

$$f'(x) = \frac{d}{dx}[f(x)] = D_x[f(x)]$$

If $y = f(x)$,

$$f'(x) = y' = \frac{dy}{dx} = \frac{d}{dx}[y]$$

At $x = x_0$

$$f'(x_0) = \frac{d}{dx}[f(x)]\bigg|_{x = x_0} = \frac{dy}{dx}\bigg|_{x = x_0}$$

$x - x_0 = \Delta x$

$f(x) - f(x_0) = f(x_0 + h) - f(x_0) = y - y_0 = \Delta y$

So,

$$f'(x) = \frac{dy}{dx} = \lim_{\Delta x \to 0}\left[\frac{\Delta y}{\Delta x}\right] = \lim_{\Delta x \to 0}\left[\frac{f(x_0 + \Delta x) - f(x)}{\Delta x}\right]$$

Finding Derivatives

1. Differentiation technique:

$$f'(x) = \lim_{h \to 0} \left[\frac{f(x+h) - f(x)}{h} \right]$$

2. The derivative of any constant function is zero

$$(c)' = 0$$

Obvious: Horizontal line has a horizontal tangent at each point

3. The Power Rule:

For any real number n

$$(x^n)' = nx^{n-1}$$

Proof for positive integers, $n = 0, 1,2, \ldots$

Recall:

$$a^2 - b^2 = (a - b)(a + b)$$

$$a^3 - b^3 = (a - b)(a^2 + ab + b^2)$$

$$a^4 - b^4 = (a - b)(a^3 + a^2b + ab^2 + b^3)$$

$$\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots$$

$$a^n - b^n = (a - b)(a^{n-1} + a^{n-2}b + \cdots + ab^{n-2} + b^{n-1})$$

$$(x^n)' = \lim_{h\to 0}\left[\frac{(x+h)^n - x^n}{h}\right] =$$

$$= \lim_{h\to 0}\left[\frac{1}{h}(x+h-x)[(x+h)^{n-1} + (x+h)^{n-2}x + \cdots + (x+h)x^{n-2} + x^{n-1}]\right] =$$

$$= \lim_{h\to 0}[(x+h)^{n-1} + (x+h)^{n-2}x + \cdots + (x+h)x^{n-2} + x^{n-1}] =$$

$$= x^{n-1} + \underbrace{x^{n-2}x}_{x^{n-1}} + \cdots + \underbrace{x \cdot x^{n-2}}_{x^{n-1}} + \underbrace{x^{n-1}}_{x^{n-1}} = nx^{n-1}$$

$$\frac{d}{dx}[x^n] = nx^{n-1}$$

Examples:

function	1. derivative
$f(x) = cx = cx^1$	$f'(x) = cx^{1-1} = c$
$f(x) = x^2$	$f'(x) = 2x^{2-1} = 2x$
$f(x) = x^3$	$f'(x) = 3x^{3-1} = 3x^2$

Constance Multiple, Sum and Difference Rules

Theorem: If f, g are differentiable at x and c is any real number
Then

$$\left(cf(x)\right)' = cf'(x)$$
$$(f(x) + g(x))' = f'(x) + g'(x)$$
$$\left(f(x) - g(x)\right)' = f'(x) - g'(x)$$

Exercise: Find $f'(x)$

$$f(x) = 2 + x^{0,5}$$

$$f(x) = 5x^2 - 3x$$

$$f(x) = \frac{6}{\sqrt{x^3}}$$

Solution:

function	1. derivative
$f(x) = 2 + x^{0,5}$	$f'(x) = 0{,}5x^{0,5-1} = 0{,}5x^{-0,5}$
$f(x) = 5x^2 - 3x$	$f'(x) = 10x - 3$
$f(x) = \frac{6}{\sqrt{x^3}} = 6x^{-\frac{3}{2}}$	$f'(x) = 6\left(-\frac{3}{2}\right)x^{-\frac{3}{2}-1} = -\frac{18}{2}x^{-\frac{5}{2}} = -\frac{9}{\sqrt{x^5}}$

The Product Rule

Observe:

$$\big(f(x)\cdot g(x)\big)' \neq f'(x)\cdot g'(x)$$

Example:

$$f(x) = 1, \qquad g(x) = x$$

$$f'(x) = 0, \qquad g'(x) = 1,$$

$$f'(x)\cdot g'(x) = 0\cdot 1 = 0$$

$$\big(f(x)\cdot g(x)\big) = x$$

$$\big(f(x)\cdot g(x)\big)' = 1 \neq f'(x)\cdot g'(x) = 0$$

Theorem: If

f, g are differentiable at x

then

$$\left(f(x) \cdot g(x)\right)' = f'(x)g(x) + f(x)g'(x)$$

Proof:

$$\left(f(x)\cdot g(x)\right)' = \lim_{h\to 0}\left[\frac{f(x+h)\cdot g(x+h) - f(x)\cdot g(x)}{h}\right] =$$

$$= \lim_{h\to 0}\left[\frac{f(x+h)\cdot g(x+h) - \overbrace{f(x+h)\cdot g(x) + f(x+h)\cdot g(x)}^{=0} - f(x)\cdot g(x)}{h}\right] =$$

$$= \lim_{h\to 0}\left[\frac{f(x+h)[g(x+h) - g(x)]}{h}\right] + \lim_{h\to 0}\left[\frac{g(x)[f(x+h) - f(x)]}{h}\right] =$$

$$= \lim_{h\to 0} f(x+h)\cdot \lim_{h\to 0}\left[\frac{g(x+h) - g(x)}{h}\right] + \lim_{h\to 0} g(x)\cdot \lim_{h\to 0}\left[\frac{f(x+h) - f(x)}{h}\right] =$$

$$= f(x)\cdot g'(x) + g(x)\cdot f'(x)$$

Sometimes we write

$$\frac{d}{dx}[uv] = u\frac{dv}{dx} + v\frac{du}{dx}$$

Generalized Product Rule:

$$\left(\prod_{i=1}^{n} f_i\right)' = (f_1 \cdot f_2 \cdot f_3 \cdot \ldots \cdot f_n)'$$
$$= f_1' \cdot f_2 \ldots f_n + f_1 \cdot f_2' \cdot \ldots \cdot f_n + \cdots + f_1 \cdot f_2 \ldots f_{n-1}' f_n + f_1 \cdot f_{n-1} \cdot f_n{}'$$

Example:

$$f(x) = 2x^3(x-1)$$

Solution:

$$f'(x) = 3 \cdot 2x^2(x-1) + 2x^3 \cdot 1 = 6x^3 - 6x^2 + 2x^3 = 8x^3 - 6x^2$$

or:

$$f'(x) = 2 \cdot 4x^3 - 2 \cdot 3x^2 = 8x^3 - 6x^2$$

The Quotient Rule

Observe:

$$\left(\frac{f(x)}{g(x)}\right)' \neq \frac{f'(x)}{g'(x)}$$

Example:

$$f(x) = 1, \qquad g(x) = x$$

$$f'(x) = 0, \qquad g'(x) = 1, \qquad \frac{f'(x)}{g'(x)} = \frac{0}{1} = 0$$

$$\frac{f(x)}{g(x)} = \frac{1}{x}$$

$$\left(\frac{f(x)}{g(x)}\right)' = -\frac{1}{x^2} \neq \frac{f'(x)}{g'(x)} = 0$$

Theorem: If f, g are differentiable at x, Then

$$\left(\frac{f(x)}{g(x)}\right)' = \frac{f'(x)g(x) - f(x)g'(x)}{g(x)^2}$$

We also write:

$$\frac{d}{dx}\left[\frac{u}{v}\right] = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$$

Handy fact:

$$\left(\frac{1}{f(x)}\right)' = \frac{f(x) \cdot 0 - 1 \cdot f'(x)}{f(x)^2} = -\frac{f'(x)}{f(x)^2}$$

Example:

$$f(x) = \frac{3x^2}{5 - x}$$

Solution

$$f^{'}(x) = \frac{3 \cdot 2x(5 - x) - 3x^2(-1)}{(5 - x)^2} = \frac{30x - 6x^2 + 3x^2}{(5 - x)^2} = \frac{-3x^2 + 30x}{(5 - x)^2}$$

The Chain Rule: Derivatives of Composition of functions

Motivating example: $f(x) = (x^2 + 1)^{100}$. Find $f'(x)$

Our only technique is to multiply this out – weary tedious.

Instead, think of $(x^2 + 1)^{100}$ as the composition of two functions.

Suppose

$$f(x) = x^{100}$$

$$g(x) = x^2 + 1$$

Then

$$f\big(g(x)\big) = f(x^2 + 1) = (x^2 + 1)^{100}$$

We can use the derivatives of x^{100} and $x^2 + 1$ to calculate the derivative $[wrt\ x]$ of

$$y = (x^2 + 1)^{100}$$

Rewrite

$$y = (x^2 + 1)^{100}$$

as

$$y = u^{100}, where\ u = x^2 + 1$$

Then

$$y'(u) = \frac{dy}{du} = 100u^{99}, and\ u'(x)\,\frac{du}{dx} = 2x$$

To get $y'(x) = \frac{dy}{dx}$ we multiply

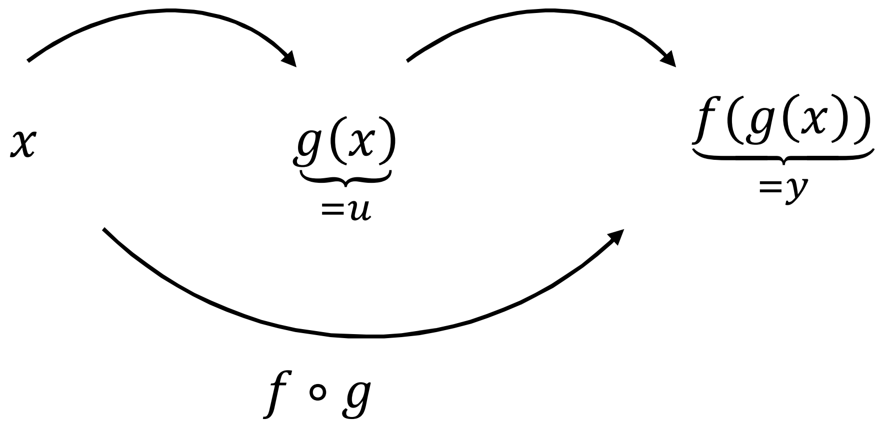
$$\frac{dy}{dx} = \underbrace{\frac{dy}{du}}_{outer} \cdot \underbrace{\frac{du}{dx}}_{inner}$$

$$y'(x) = y'(u) \cdot u'(x) = 100(x^2 + 1)^{99} \cdot 2x = 200x \cdot (x^2 + 1)^{99}$$

Theorem [The "Chain" Rule]

If g is differentiable at x and f is differentiable at $g(x) = u$

Then $y = (f \circ g)(x)$ is differentiable at x



Exercise: Find $f(x)$

$$f(x) = 4(x^2 - 1)^2$$

Solution:

$$f(x) = 4\underbrace{(x^2 - 1)}_{u}^2$$

$$y(u) = 4u^2; \quad u(x) = x^2 - 1$$

$$f^{'}(x) = y(u) \cdot u(x) = 4 \cdot 2(x^2 - 1)2x = 16x^3 - 16x = 16x(x^2 - 1)$$

Derivatives of Trigonometric Functions

Recall:

$$\lim_{h\to 0}\left[\frac{sinh}{h}\right] = 1;\ \lim_{h\to 0}\left[\frac{1-cosh}{h}\right] = 0$$

Then

$$sin'x = \lim_{h\to 0}\left[\frac{\sin(x+h) - sinx}{h}\right] =$$

$$= \lim_{h\to 0}\left[\frac{\sin xcosh + cosxsinh - sinx}{h}\right] =$$

$$= \lim_{h\to 0}\left[sinx\left(\frac{cosh-1}{h}\right) + cosx\left(\frac{sinh}{h}\right)\right] =$$

$$= \lim_{h\to 0}\left[sinx\underbrace{\left(\frac{cosh-1}{h}\right)}_{\to 0}\right] + \lim_{h\to 0}\left[cosx\underbrace{\left(\frac{sinh}{h}\right)}_{\to 1}\right] = cosx$$

$$[sinx]' = cosx$$

$$[cosx]' = -sinx$$

$$[tanx]' = \frac{1}{cos^2x}$$

$$[cotx]' = -\frac{1}{sin^2x}$$

Exercises: Find $f'(x)$

function	1. derivative
$f(x) = cos2x$	$f'(x) = -2sin2x$
$f(x) = \sin(x^2)$	$f'(x) = 2xcos(x^2)$
$f(x) = cos^2x$	$f'(x) = 2cosx(-sinx)$

Derivatives of Inverse Trigonometric Functions

$$[arcsinx]' = \frac{1}{\sqrt{1-x^2}}$$

$$[arccosx]' = -\frac{1}{\sqrt{1-x^2}}$$

$$[arctanx]' = \frac{1}{1+x^2}$$

$$[arccotx]' = -\frac{1}{1+x^2}$$

Derivatives Involving Logarithms

We find

$$(lnx)' = \frac{d}{dx}[lnx]$$

For $x > 0$ [Domain of lnx]

We need two facts to recall:

1. lnx is continuous

So, at any a we have:

$$lna = \lim_{x \to a}[lnx] = ln\left[\lim_{x \to a} x\right]$$

The limit “moves through the ln

2. Definition: "e" is that number which $\left(1+\frac{1}{n}\right)^n$ approaches, as $n \to \infty$

$e \approx 2{,}71828$ ...

$$\lim_{\substack{x \to -\infty \\ x \to +\infty}} \left[\left(1+\frac{1}{x}\right)^x\right] = e$$

Let $u = \frac{1}{x}$, so $\begin{matrix} x \to +\infty \; means \; u \to 0^+ \\ x \to +\infty \; means \; u \to 0^- \end{matrix}$

Thus

$$\lim_{u \to 0} \left[(1+u)^{\frac{1}{u}}\right] = e$$

[Limit is two-sided]

So,

$$(lnx)' = \lim_{h\to 0}\left[\frac{\ln(x+h) - lnx}{h}\right] = \lim_{h\to 0}\left[\frac{1}{h}\ln\left(\frac{x+h}{x}\right)\right] =$$

$$= \lim_{h\to 0}\left[\frac{1}{h}\ln\left(1+\frac{h}{x}\right)\right]$$

Let $v = \frac{h}{x}$ so $v \to 0, h \to 0$

$$= \lim_{v\to 0}\left[\frac{1}{vx}\ln(1+v)\right] = \frac{1}{x}\cdot\lim_{v\to 0}\left[\ln(1+v)^{\frac{1}{v}}\right] =$$

$$\frac{1}{x}\cdot\ln\underbrace{\left[\lim_{v\to 0}(1+v)^{\frac{1}{v}}\right]}_{=e} = \frac{1}{x}\underbrace{lne}_{=1} = \frac{1}{x}$$

So,

$$(lnx)' = \frac{d}{dx}[lnx] = \frac{1}{x}, \qquad x > 0$$

Generalized version:

$$\frac{d}{dx}[lnu] = \frac{1}{u}\frac{du}{dx}, \qquad u(x) > 0$$

So,

$$(lnx)' = \frac{1}{x}, \qquad for\ x > 0$$

$$(\log_b x)' = \frac{1}{lnb}\frac{1}{x},\ for\ x > 0\ [\log_b x = \frac{lnx}{lnb}]$$

Exercises:

function	1. derivative
$f(x) = ln(2x-1)$	$f'(x) = \frac{1}{2x-1} \cdot 2 = \frac{1}{x-0{,}5}$
$f(x) = \frac{1}{lnx} = (lnx)^{-1}$	$f'(x) = -(lnx)^{-2} \cdot \frac{1}{x} = -\frac{1}{x(lnx)^2}$
$f(x) = xln(3-x^2)$	$f'(x) = ln(3-x^2) - \frac{2x^2}{(3-x^2)}$

Derivatives of Exponential Functions

What is

$$(b^x)' = \frac{d}{dx}[b^x],$$

$$b \geq 0,$$

$$b \neq 0,$$

$$b \neq 1$$

$$?$$

Development:

$$u = b^x$$

$$lnu = xlnb$$

$$\frac{d}{dx}\left[\underbrace{lnu}_{=y}\right] = \frac{d}{dx}[xlnb]$$

$$\frac{1}{u}\frac{du}{dx} = lnb \cdot 1$$

$$\frac{du}{dx} = ulnb$$

Since

$$u = b^x$$

$$\frac{d}{dx}[b^x] = (b^x)' = b^x lnb$$

Important case:

If $b = e$

$$(e^x)' = e^x lne = e^x$$

Exercises:

function	1. derivative
$f(x) = e^{5x}$	$f'(x) = 5e^{5x}$
$f(x) = \frac{e^{5x}}{x^2} = e^{5x} \cdot x^{-2}$	$f'(x) = 5e^{5x} \cdot x^{-2} + e^{5x}(-2)x^{-3} = \frac{e^{5x}(5x-2)}{x^3}$
$f(x) = \sqrt{e^{2x} + x}$	$f'(x) = \frac{1}{2}(e^{2x} + x)^{-\frac{1}{2}} \cdot (2e^{2x} + 1)$
$f(x) = 2^x$	$f'(x) = (ln2)2^x$
$f(x) = 2^{3x}$	$f'(x) = 3(ln2)2^{3x}$
$f(x) = x \cdot 2^{3x}$	$f'(x) = 2^{3x} + 3x(ln2)2^{3x}$

Notation for Derivatives of Derivatives [Higher order Derivatives]

1st Derivative:

$$f'(x),\ \ \frac{d}{dx}[f(x)],\ \ y',\ \ \frac{d}{dx}[y]=\frac{dy}{dx}$$

2nd Derivative:

$$f''(x),\ \ \frac{d}{dx}\left[\frac{d}{dx}f(x)\right]=\frac{d^2}{dx^2}[f(x)],\ \ y'',\ \ \frac{d}{dx}\left[\frac{d}{dx}(y)\right]=\frac{d^2y}{dx^2}$$

The second derivative of y wrt x

For higher derivatives

$$f^{(n)}(x),\ \ \frac{d^ny}{dx^n}=\frac{d^n}{dx^n}[f(x)]$$

The differentiations rules are the same

Exercise:

$$f(x) = 3x^4 - 2x^3 + x^2 - 4x + 2$$

$$f'(x) = 12x^3 - 6x^2 + 2x - 4$$

$$f''(x) = 36x^2 - 12x + 2$$

$$f^{(3)}(x) = 72x - 12$$

$$f^{(4)}(x) = 72$$

$$f^{(n)}(x) = 0 \ for \ all \ n = 5,6,7 \ldots$$