Applications of derivatives:
Extremal problems in two variables

Local maxima and minima

At a local max or min, $f_x = 0$ and $f_y = 0$

Definition of a critical point: (x_0, y_0) where $f_x = 0$ and $f_y = 0$

A critical point may be a local minimum, local maximum, or saddle.

Second derivative test

Goal: determine type of a critical point, and find the local min/max.
Note: local min/max occur at critical points

General case: second derivative test.
We look at second derivatives:

$$f_{xx} = \frac{\partial^2 f}{\partial x^2};\ f_{xy} = \frac{\partial^2 f}{\partial x \partial y} = f_{yx} = \frac{\partial^2 f}{\partial y \partial x};\ f_{yy} = \frac{\partial^2 f}{\partial y^2}$$

The **Hessian matrix** (or simply the **Hessian**) is the square matrix of second-order partial derivatives of a function

$$H(f) = \begin{pmatrix} \frac{\partial^2 f}{\partial x^2} & \frac{\partial^2 f}{\partial x \partial y} \\ \frac{\partial^2 f}{\partial x \partial y} & \frac{\partial^2 f}{\partial y^2} \end{pmatrix} = \begin{pmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{pmatrix}$$

Given is f and a critical point (x_0, y_0).

Define the second derivative test discriminant as

$$D = f_{xx}(x_0, y_0) \cdot f_{yy}(x_0, y_0) - f_{xy}(x_0, y_0) \cdot f_{yx}(x_0, y_0)$$

Then

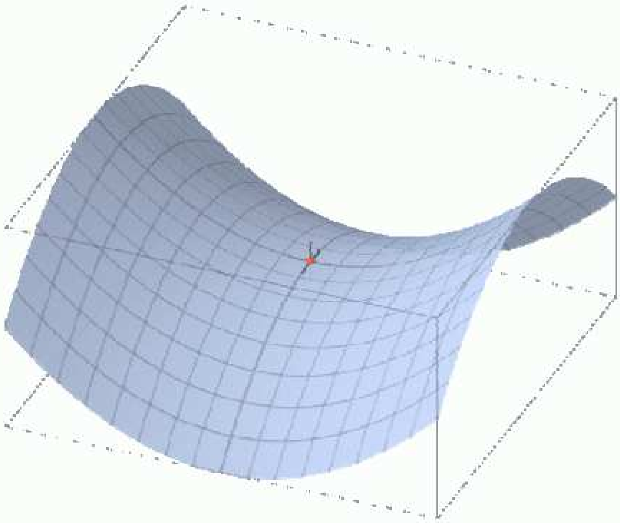
If $D > 0$ and $f_{xx}(x_0, y_0) > 0$ $\Longrightarrow$ **local minimum**

If $D > 0$ and $f_{xx}(x_0, y_0) < 0$ $\Longrightarrow$ **local maximum**

If $D < 0$ $\Longrightarrow$ **saddle**

If $D = 0$ $\Longrightarrow$ **cannot be concluded**

A **saddle point** is a point in the range of a function that is a **critical point** but not a local extremum. The name derives from the fact that the prototypical example in two dimensions is a surface that **curves up** in one direction, and **curves down** in a different direction, resembling a saddle or a mountain pass.



http://en.wikipedia.org/wiki/Saddle_point

Example:

$$f(x,y) = y^3 + x^2(y+1) - 12y + 11$$

$$f_x = (y+1)2x \qquad f_y = 3y^2 + x^2 - 12$$

$$f_{xx} = 2y + 2 \qquad f_{yy} = 6y \qquad f_{yx} = f_{xy} = 2x$$

Critical points candidates: First derivative test applied

$$f_x = (y+1)2x = 0 \quad f_y = 3y^2 + x^2 - 12 = 0$$

We need to solve the following system of equations:

$$\begin{cases} (y+1)2x = 0 \\ 3y^2 + x^2 - 12 = 0 \end{cases}$$

The critical points are:

$$(x_1, y_1) = (3, -1)\ ;\ (x_2, y_2) = (-3, -1);\ (x_3, y_3) = (0, -2)\ ;\ (x_4, y_4) = (0,2)$$

Maximum, minimum or saddle? Second derivative test applied:

$$f_{xx} = 2y + 2 \quad f_{yy} = 6y;\ f_{yx} = f_{xy} = 2x$$

$(\boldsymbol{x_1}, \boldsymbol{y_1}) = (\mathbf{3}, -\mathbf{1})$

$$f_{xx}(x_0,y_0) \cdot f_{yy}(x_0,y_0) - f_{xy}(x_0,y_0) \cdot f_{yx}(x_0,y_0) = 0 - 36 = -36 < 0\ \boldsymbol{saddle}$$

$(\boldsymbol{x_2}, \boldsymbol{y_2}) = (-\mathbf{3}, -\mathbf{1})$

$$f_{xx}(x_0,y_0) \cdot f_{yy}(x_0,y_0) - f_{xy}(x_0,y_0) \cdot f_{yx}(x_0,y_0) = 0 - 36 = -36 < 0\ \boldsymbol{saddle}$$

$(\boldsymbol{x_3}, \boldsymbol{y_3}) = (\mathbf{0}, -\mathbf{2})$

$$f_{xx}(x_0,y_0) \cdot f_{yy}(x_0,y_0) - f_{xy}(x_0,y_0) \cdot f_{yx}(x_0,y_0) = 24 - 0 = 24 > 0$$

$$f_{xx}(x_0,y_0) = -2 < 0\ \boldsymbol{maximum}$$

$(\boldsymbol{x_4}, \boldsymbol{y_4}) = (\mathbf{0}, \mathbf{2})$

$$f_{xx}(x_0,y_0) \cdot f_{yy}(x_0,y_0) - f_{xy}(x_0,y_0) \cdot f_{yx}(x_0,y_0) = 72 - 0 = 72 > 0$$

$$f_{xx}(x_0,y_0) = 6 > 0\ \boldsymbol{minimum}$$

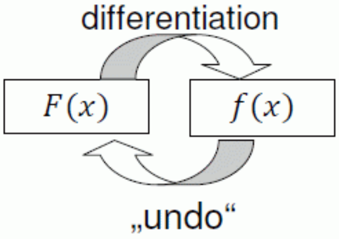
Integration

The Integral of a Function. The Indefinite Integral

Undoing a derivative: **Antiderivative** = Indefinite Integral

Definition: A function $F(x)$ is called an **antiderivative** of a function $f(x)$ on same interval $I = [a, b]$, if

$$F'(x) = f(x)$$
$$for\ all\ x\ in\ I$$



Note: Unlike derivatives, antiderivatives **are not unique:**

Example:

$F(x) = \frac{1}{3}x^3$ is an antiderivative of $f(x) = x^2$ on $(-\infty, \infty)$

because

$$F'(x) = \frac{d}{dx}\left[\frac{1}{3}x^3\right] = x^2 = f(x)$$

But also for any constant c

$$\frac{d}{dx}\left[\frac{1}{3}x^3 + c\right] = x^2 = f(x)$$

because

$$\frac{d}{dx}[c] = 0$$

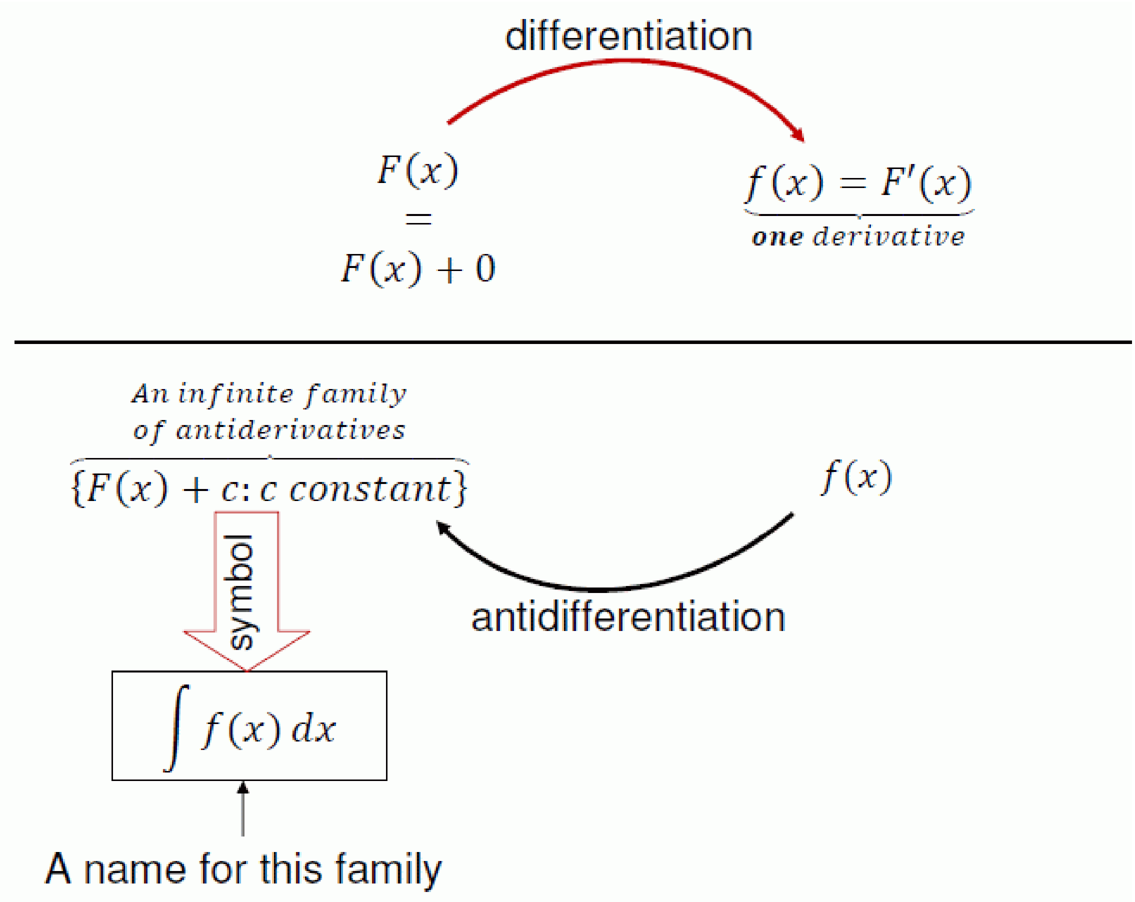
Theorem:

If $F(x)$ is any antiderivative of $f(x)$ on I,

then so is $F(x) + c \leftarrow any\ constant$

Every antiderivative of $f(x)$ on I has the form $\boxed{F(x) + c}$ for some c

- Differentiation produces **one** derivative
- Antidifferentiation produces **an infinite family** of antiderivatives

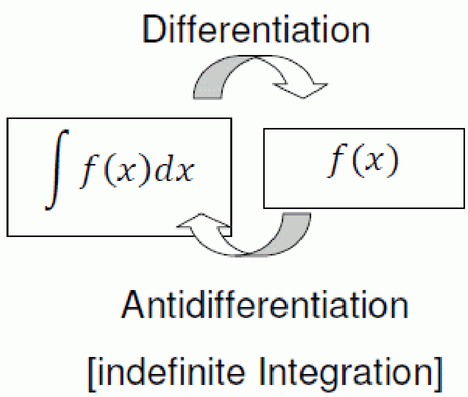


$$\underbrace{\int f(x)dx} = F(x) + c$$

The indefinite integral of $f(x)$

- $\int$ – the integral sign [elongated “S”]
- $f(x)$ - the integrand
- dx - indicates the independent variable
- c - constant of integration
- *F*(*x*) + *c* - one of many antiderivatives of *f*

The indefinite integral of *f* represents the entire family of all antiderivatives of *f*.



$$\frac{d}{dx}\left[\int f(x)dx\right] = f(x)$$

Note: Sometimes we write:

$$\int 1dx \ as \int dx$$

$$\int \frac{1}{x^2}dx \ as \int \frac{dx}{x^2}$$

Finding Antiderivatives

(1) Use derivatives we know to build a table

Derivative	Corresponding antiderivative
$\frac{d}{dx}[x] = 1$	$\int 1dx = x + c$
$\frac{d}{dx}\left[\frac{x^{r+1}}{r+1}\right] = x^r$ where $r \neq -1$	$\int x^r\,dx = \left[\frac{x^{r+1}}{r+1}\right] + c$ "Add 1 to the power and divide by this new power"
$\frac{d}{dx}[sinx] = cosx$	$\int cosxdx = sinx + c$
$\frac{d}{dx}[cosx] = -sinx$	$\int sinxdx = -\,cosx + c$
$\frac{d}{dx}[tanx] = \frac{1}{cos^2x}$	$\int \frac{1}{cos^2x}dx = tanx + c$
$\frac{d}{dx}[cotx] = -\frac{1}{sin^2x}$	$\int \frac{1}{sin^2x}dx = -\,cotx + c$
$\frac{d}{dx}[arcsinx] = \frac{1}{\sqrt{1-x^2}}$	$\int \frac{1}{\sqrt{1-x^2}}dx = -\,arcsinx + c$
$\frac{d}{dx}[arctanx] = \frac{1}{1+x^2}$	$\int \frac{1}{1+x^2}dx = arctanx + c$
$\frac{d}{dx}[arccotx] = -\frac{1}{1+x^2}$	$\int \frac{1}{1+x^2}dx = -\,arctanx + c$
$\frac{d}{dx}[e^x] = e^x$	$\int e^x dx = e^x + c$
$\frac{d}{dx}[a^x] = a^x lna$	$\int a^x dx = \frac{a^x}{lna} + c$
$\frac{d}{dx}[lnx] = \frac{1}{x}$	$\int \frac{1}{x}dx = lnx + c$
$\frac{d}{dx}[\log_a x] = \frac{1}{xlna}$	$\frac{1}{lna}\int \frac{1}{x}dx = \frac{lnx}{lna} + c = \log_a x + c$
$\int lnxdx = x(lnx) - x + c$	
$\int \log_a x\,dx = \frac{1}{lna}(x \cdot (lnx) - x) + c$	

(2) Some Properties on Indefinite Integrals: c a real number

$$\int cf(x)dx = c\int f(x)dx$$

$$\int [f(x) + g(x)]dx = \int f(x)dx + \int g(x)dx$$

$$\int [f(x) - g(x)]dx = \int f(x)dx - \int g(x)dx$$

All applied earlier for limits + derivatives

Do not write:

$$\int 2xdx = 2\int xdx = 2\left(\frac{x^2}{2} + c\right) = x^2 + \cancel{2c} = x^2 + c$$

$$\int (1 + x)dx = \int 1dx + \int xdx = (x + \cancel{c_1}) + \left(\frac{x^2}{2} + \cancel{c_2}\right) = x + \frac{x^2}{2} + c$$

Notes on constants of integration:

- Do not forget the constant of integration!
- Do not introduce it too soon
- Combine multiple constants of integration into one *c*

Integration techniques considered so far:

(1) Use (create) a table

(2) Rewrite the integrand (in order to use the table)

Examples:

$$\int 2\cdot x^2dx = 2\cdot\int x^2dx = \frac{2}{3}x^3 + c$$

$$\int (x^2 + 3sinx)dx = \int x^2dx + \int 3sinxdx = \frac{x^3}{3} - 3cosx + c$$

The Indefinite Integration by Parts

$$\int f(x) \cdot g(x)\, dx = ?$$

Recall the product rule for derivatives $u = u(x)$, $v = v(x)$

$$[u(x) \cdot v(x)]' = u'(x) \cdot v(x) + u(x) \cdot v'(x)$$

$$u'(x) \cdot v(x) = [u(x) \cdot v(x)]' - u(x) \cdot v'(x)$$

Integrate both sides

$$\int u'(x) \cdot v(x)\, dx = \int [u(x) \cdot v(x)]' dx - \int u(x) \cdot v'(x)\, dx$$

$$\int u'(x) \cdot v(x) = u(x) \cdot v(x) - \int u(x) \cdot v'(x)\, dx$$

Shorthand notation: The integration by part formula

$$\int v du = uv - \int u dv$$

Generally try to choose v to be something that simplifies when you differentiate it.

Integration by parts formula: $\int u'(x) \cdot v(x) = u(x) \cdot v(x) - \int u(x) \cdot v'(x)\, dx$

Example 1: $$\int 2xe^x dx$$

How to choose u and v?
$u'(x) = 2x \quad v(x) = e^x$

$u(x)$ and $v'(x)$ are easy to find: $u(x) = x^2$ und $v'(x) = e^x$

But we cannot find the indefinite Integral of the product $u(x)v'(x) = x^2 \cdot e^x$

Then:

$u'(x) = e^x \quad v(x) = 2x$

$u(x) = e^x$ and $v'(x) = 2$, so $\int u(x)v'(x)dx = 2e^x$

$$\int e^x 2x dx = 2xe^x - 2\int e^x dx = e^x(2x - 2) + c$$

Integration by parts formula: $\int u'(x) \cdot v(x) = u(x) \cdot v(x) - \int u(x) \cdot v'(x)\, dx$

Example 2:

$$\boxed{\int e^x x^2 dx}$$

$$u'(x) = e^x \quad v(x) = x^2$$

$$u(x) = e^x \quad v'(x) = 2x$$

$$\int e^x x^2 dx = e^x x^2 - \int (e^x \cdot 2x) dx =$$

$$= \; e^x x^2 - \left(2e^x x - 2\int (e^x \cdot 1) dx\right)$$

$$= \; e^x x^2 - 2xe^x - 2e^x = e^x(x^2 - 2x + 2) + c$$

Integration by parts formula: $\int u'(x) \cdot v(x) = u(x) \cdot v(x) - \int u(x) \cdot v'(x)\, dx$

Example 3:

$$\boxed{\int cosx \cdot sinxdx}$$

$$\int \underbrace{cosx}_{u'(x)} \cdot \underbrace{sinx}_{v(x)} dx = sinx \cdot sinx - \int sinx \cdot cosxdx$$

$$2\int cosx \cdot sinxdx = sin^2 x$$

$$\int cosx \cdot sinxdx = \frac{1}{2} sin^2 x$$

The Indefinite Integration by Substitution

Idea: Suppose $F' = f$ and g' exists

Chain rule:

$$F'(g(x)) = \underbrace{F'(g(x))}_{outer} \cdot \underbrace{g'(x)}_{inner}$$

$$\int F'(g(x)) \cdot g'(x)\, dx = \int F'(g(x))$$

So,

$$\int f(g(x)) \cdot g'(x)\, dx = F(g(x)) + c$$

Let $u = g(x)$, then:

$$f(g(x)) = \boldsymbol{f(u)}$$

$$\frac{du}{dx} = g'(x) \;\; \rightarrow \;\; g'(x)dx = \boldsymbol{du}$$

$$\int f(u)\, du = F(u) + c$$

Substitution of u for $g(x)$ makes (when it works!) integration easier.

Application of the substitution technique:
Always consider “substitution” first.
If one substitution fails, try another one!

Always make a total change from one variable (*x*) to another (*u*). Never mix variables!

Key requirement for applying substitution:

Find something in the integrand to call u to simplify the appearance of the integral and whose $du = \frac{du}{dx}dx$ is also present as a factor.

Example:

$$\int \sqrt{1+x}dx$$

$$u = 1 + x$$

$$\frac{du}{dx} = 1 \rightarrow dx = du$$

$$\int \sqrt{u}du = \frac{2}{3}u^{\frac{3}{2}} = \frac{2}{3}(1+x)^{\frac{3}{2}}$$

Exercises:

function	substitution	Integral
$f(x) = e^{2x}$	$u = 2x$	$F(x) = \frac{1}{2}e^{2x} + c$
$f(x) = (x+1)^2$	$u = x + 1$	$F(x) = \frac{1}{3}(x+1)^3 + c$
$f(x) = xln(x^2)$	$u = x^2$	$F(x) = \frac{1}{2}(x^2ln(x^2) - x^2) + c$

Summary

A hard and fast set of rules for determining the method that should be used for integration does not exist.

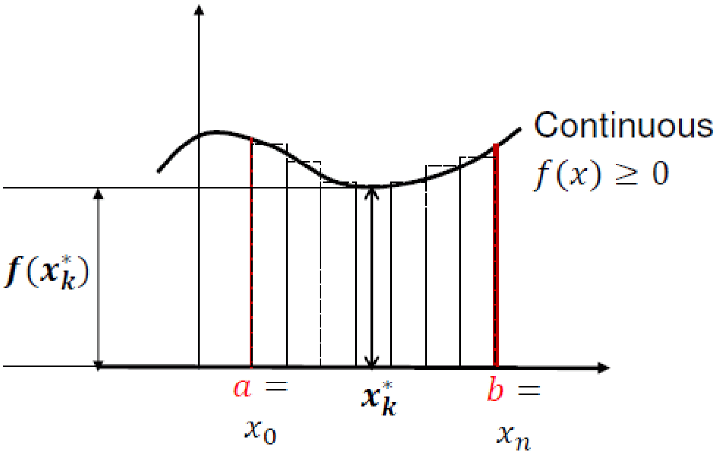
Some integrals can be done in more than one way.

It is possible that you will need to use more than one method to compute an integral.

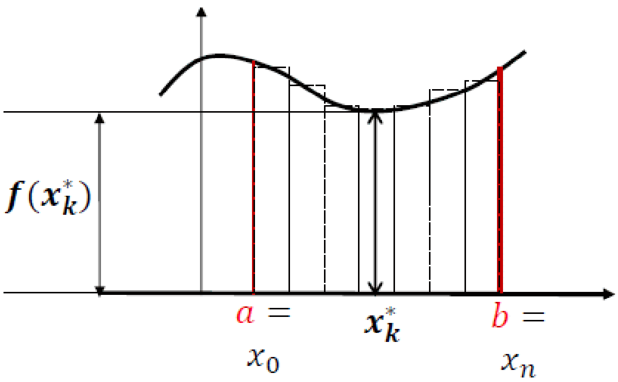
There are integrals that cannot be computed in terms of functions that we know.

Application of integration: Calculation of areas

Definition of Area "under a Curve"



- Partition into n **equal** subintervals
- Each width $= \frac{1}{n}(b-a) = \Delta x$



- Choose any point in each interval to calculate rectangle heights

$$\begin{bmatrix} Area\ under \\ Curve \end{bmatrix} \approx \sum_{k=1}^{n} \left[\underbrace{f(x_k^*)\Delta x}_{\substack{Area\ of\ one \\ rectangle}} \right]$$

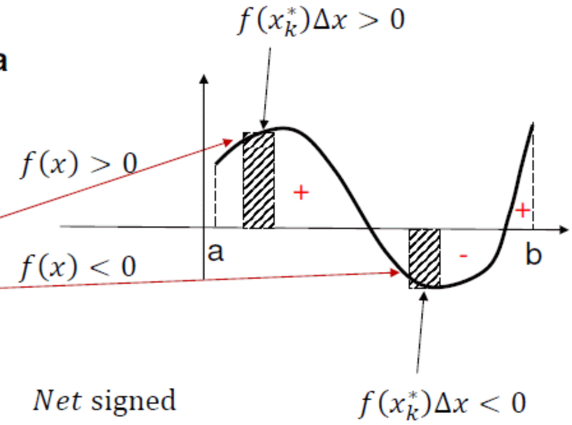
Definition: If f is continuous on $[a, b]$ $f(x) \geq 0$ on $[a, b]$

Then

$$\begin{bmatrix} Area\ under \\ y = f(x) \\ over\ [a, b] \end{bmatrix} = \lim_{n\to\infty} \sum_{k=1}^{n} f(x_k^*)\Delta x$$

Net Area

Definition: Net „Signed Area“



If f is continuous on $[a,b]$ $f(x) \geq 0$ Then

$$\lim_{n\to\infty}\left[\sum_{k=1}^{n} f(x_k^*)\Delta x\right] = \begin{matrix} \textit{Net signed} \\ \textit{Area between} \\ y = f(x) \textit{ and } [a,b] \end{matrix}$$

Approximating Area Numerically

For **large** n

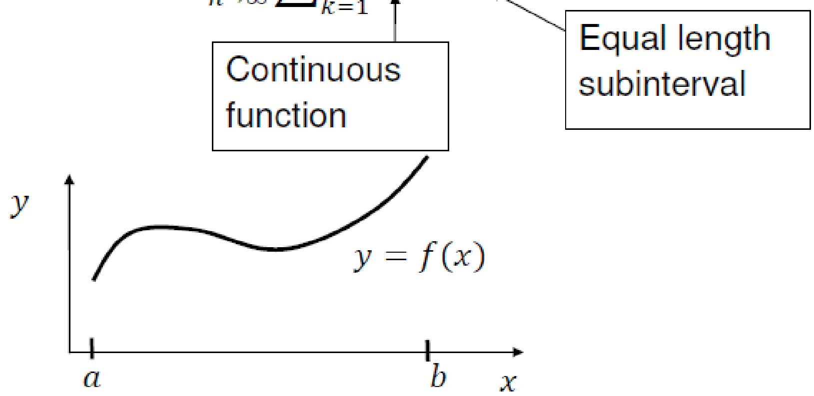
$$\lim_{n\to\infty}\sum_{k=1}^{n} f(x_k^*)\Delta x \approx \sum_{k=1}^{n} f(x_k^*)\Delta x$$

The Definite Integral

The Definite Integral Defined

Extend our “Net Area” limit:

$$\lim_{n\to\infty}\sum_{k=1}^{n} f(x_k^*)\Delta x$$

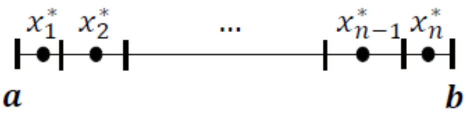


To compute the area under the graph of $f(x)$ and above the interval $[a,b]$ we proceed as follows:

1. Subdivide the interval $[a, b]$ into n **unequal** subintervals with endpoints:

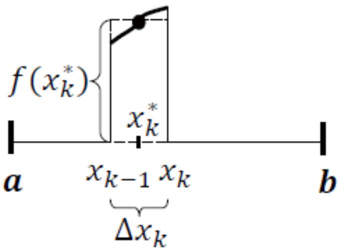
$$a = x_0 < x_1 < x_2 < \cdots < x_{n-2} < x_{n-1} < x_n = b$$

For each $k = 1,2, \ldots n-1, n$ let $\Delta x_k = x_k - x_{k-1} = lenght\ of\ [x_{k-1}, x_k]$



Note: The largest of the Δx_k will be denoted $\Delta \boldsymbol{x_{max}}$

2. Inside each $[x_{k-1}, x_k]$select a point x_k^*, evaluate $f(x_1^*), f(x_2^*), \ldots, f(x_{n-1}^*), f(x_n^*)$ and compute $f(x_1^*)\Delta x_1, f(x_2^*)\Delta x_2, \ldots, f(x_{n-1}^*)\Delta x_{n-1}, f(x_n^*)\Delta x_n$



3. Form the **Riemann Sum**. A Riemann sum is a summation of a large number of small partitions of a region.

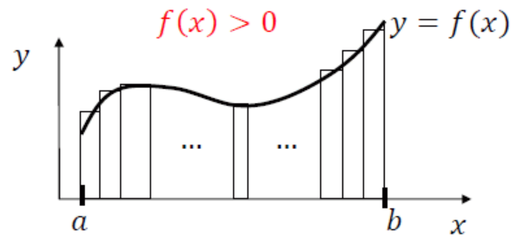
$$f(x_1^*)\Delta x_1 + f(x_2^*)\Delta x_2 + \cdots + f(x_{n-1}^*)\Delta x_{n-1} + f(x_n^*)\Delta x_n = \sum_{k=1}^{n} f(x_k^*)\Delta x_k$$

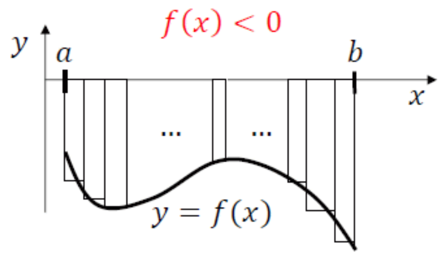
4. Repeat Step 1-3 over and over with finer and finer subdivision of $[a, b]$ (i.e. smaller and smaller Δx_{max} and take a limit

$$\lim_{\Delta x_{max} \to 0} \sum_{k=1}^{n} f(x_k^*)\Delta x_k$$

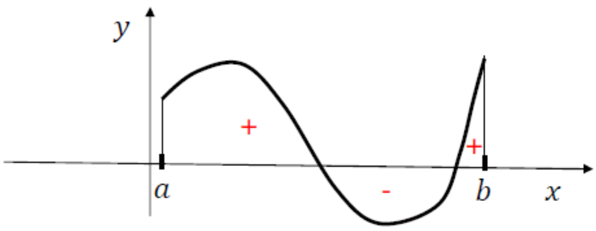
Partition in *equal* subintervals: $n \to \infty$ means $\Delta x \to 0$, which guarantees that each width shrinks
Partition in *unequal* subintervals: $\max \Delta x_k \to 0$ guarantees that each width shrinks.

Notice that if $f(x) \leq 0$ on $[a,b]$, then the result of this procedure will be minus the area between the graph of $f(x)$ and $[a,b]$.





If $f(x)$ takes both positive and negative value on $[a,b]$, then the procedure yields the net signed area between the graph of $f(x)$ and the interval $[a,b]$



Definite Integral: Definition

1. f is integrable on $[a,b]$ if $\lim_{max\Delta x_k \to 0} \sum_{k=1}^{n} f(x_k^*)\Delta x_k$ (Riemann Sum)

exists and does not depend on

- the choice of partition
- or the choice of $x_k{}^*$ point

2. If f is integrable, then the limit

$$\lim_{max\Delta x_k \to 0} \sum_{k=1}^{n} f(x_k^*)\Delta x_k$$

is called the **Definite Integral** of $f(x)$ over $[a,b]$ [or from a to b] and is denoted

$$\int_a^b f(x)dx$$

a: lower limit of integration

b: upper limit of integration

Be careful not to confuse $\int_a^b f(x)dx$ and $\int f(x)dx$. They are **entirely different types** of things. The first is a **number**, the second is a collection of functions.

Notation:

$$\Delta \to d$$

$$\Delta x \to dx$$

$$\Sigma \quad \to \int$$

The definite Integral of a continuous Function = Net “Area” under a curve

Theorem: If f is continuous on $[a, b]$

then f is integrable on $[a, b]$

And

$$\begin{matrix} Net\ \pm Area \\ between\ the \\ graph\ of\ f \\ and\ [a,b] \end{matrix} = \int_a^b f(x)dx$$

Notation:

$$\int_{x=a}^{x=b} [integrand]dx$$

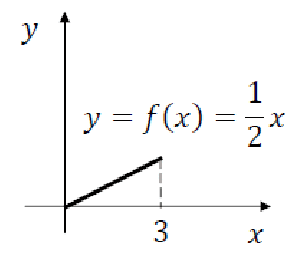
We will need methods for evaluating the number

$$\int_a^b f(x)dx$$

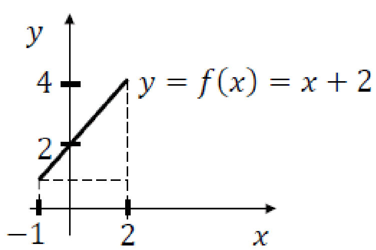
other than computing the limit that defines them.

Some methods generally involve antidifferentiation, but some definite integrals can be evaluated by thinking of them as area.

Definite Integrals Using Geometry



$$\int_0^3 \frac{1}{2}xdx = \frac{1}{2}3\left(\frac{1}{2}\cdot 3\right) = \frac{9}{4}$$



$$\int_{-1}^{2}(x+2)dx = \frac{1}{2}3\cdot 3 + 3\cdot 1 = \frac{9}{2} + 3 = \frac{15}{2}$$

Finding Definite Integrals: A new Definition and Properties

1. If a is in Domain of f, define

$$\int_a^a f(x)dx = 0$$

2. If f is integrable on $[a, b]$, define

$$\int_b^a f(x)dx = -\int_a^b f(x)dx$$

Properties of definite integrals:

$$\int_a^b [cf(x)]dx = c\int_a^b f(x)dx$$

$$\int_a^b [f(x) + g(x)]dx = \int_a^b f(x)dx + \int_a^b g(x)dx$$

$$\int_a^b [f(x) - g(x)]dx = \int_a^b f(x)dx - \int_a^b g(x)dx$$

Theorem: If f is integrable on any closed Interval containing a, b, c

Then

$$\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx$$

No matter, how a, b, c are ordered!

Theorem: Suppose, f, g integrable on $[a, b]$

a. If $f(x) \geq 0$ for all x in $[a, b]$, Then

$$\int_a^b f(x)dx \geq 0$$

b. If $f(x) \geq g(x)$ for all x in $[a, b]$, Then

$$\int_a^b f(x)dx \geq \int_a^b g(x)dx$$

The Fundamental Theorem of Calculus

There are two parts to this.

Fundamental Theorem of Calculus, part I:

If f is continuous on $[a,b]$ and $F(x)$ is any antiderivative for $f(x)$ on $[a,b]$.

then

$$\int_a^b f(x)dx = F(x)|_a^b = \underbrace{F(b)}_{upper} - \underbrace{F(a)}_{lower}$$

Notice. If F is any antiderivative of f,

$$\int_a^b f(x)dx = [F(x)+c]_a^b = [F(b)+c] - [F(a)+c] = F(b) - F(a)$$

So, we can always omit writing c here. Thus

$$\int_a^b f(x)dx = F(x)|_a^b$$

Part II:

If f is continuous on the Interval I, then f has an antiderivative on I

If a is in I then

$$F(x) = \int_a^x f(t)dt$$

is one such antiderivative for $f(x)$

meaning

$$\frac{d}{dx}\left[\int_a^x f(t)dt\right] = f(x)$$

Differentiation and Integration are Inverse Processes:

FTC, PartI

$$\int_a^x f'(t)dt = f(x) - f(a)$$

"Integral of derivative recovers original function"

FTC, PartII

$$\frac{d}{dx}\left[\int_a^x f(t)dt\right] = f(x)$$

"Derivative of integral recovers original function".

Definite and Indefinite Integrals Related:

$\int f(x)dx$	$\int_a^b f(x)dx$
is a function in x	is a number – no x involved!

So, the variable of integration in a definite integral doesn't matter: The name of the variable is irrelevant. For this reason the variable in a definite integral is often referred to as dummy variable, place holder.

$$\int_a^b f(x)dx = \int_a^b f(t)dt = \int_a^b f(y)dy$$

Some Examples:

1.

$$\int_4^4 2xdx = x^2|_4^4 = 4^2 - 4^2 = 0$$

2.

$$\int_1^2 2xdx = x^2|_1^2 = 2^2 - 1^2 = 3$$

$$-\int_2^1 2xdx = -x^2|_2^1 = -1^2 + 2^2 = 3$$

3.

$$\int_1^4 2xdx = x^2|_1^4 = 4^2 - 1^2 = 15$$

$$\int_1^2 2xdx + \int_2^4 2xdx = x^2|_1^2 + x^2|_2^4 = 2^2 - 1^2 + 4^2 - 2^2 = 15$$

Definite Integration by Substitution.

Extending the Substitution Method of Integration to definite Integrals to evaluate the number

$$\int_a^b f\left(g(x)\right)g'(x)dx \qquad \begin{matrix} g' \, continuous \; on \; [a,b] \\ f \; continuous \; where \; g \; exists \; on \; [a,b] \end{matrix}$$

Substitution:

$$u = g(x)$$

$$du = g'(x)dx$$

Change x - limits to u -limits with the substitution:

$$u(a) = g(a)$$
$$u(b) = g(b)$$

To get

$$\int_{g(a)}^{g(b)} f\,(u)du$$

Examples:

1. Find

$$\int_{-1}^{1} e^{2x}dx$$

1. x substitution of x: $u(x) = 2x = u \qquad \frac{du}{dx} = 2 \qquad dx = \frac{1}{2}du$

2. limits substitution:

lower limit: $u(-1) = -2$

upper limit: $u(1) = 2$

$$\frac{1}{2}\int_{-2}^{2} e^u = \frac{1}{2}(e^2 - e^{-2})$$

2. Find:

$$\int_{1}^{2} 2x\ln x^2\, dx$$

1. x substitution: $u(x) = x^2 = u \qquad \frac{du}{dx} = 2x \qquad dx = \frac{1}{2x}du$

2. limits substitution:

lower limit: $u(1) = 1$

upper limit: $u(2) = 4$

$$\int_{1}^{4} \ln u du = u\ln u - u|_1^4 = (4\ln 4 - 4) - (\ln 1 - 1) = 4\ln 4 - \ln 1 - 3$$

The Definite Integral Applied

Total Area

Although

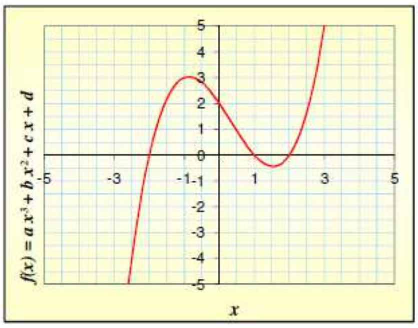
$$\int_a^b f\,(x)dx - \text{"net area"}$$

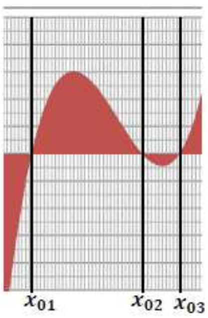
We can find that

$$\begin{bmatrix} total \\ Area \end{bmatrix} = \int_a^b |f(x)|\, dx$$

Example. Compute the area between $(x) = 0{,}5x^3 - 0{,}5x^2 - 2x + 2$, the x –axis and the lines $x_1 = -2{,}5$ and $x_2 = 2{,}5$:

Nullpoints: $\quad f(x) = 0{,}5x^3 - 0{,}5x^2 - 2x + 2 = 0{,}5(x + 2)(x - 1)(x - 2)$





$x_{01} = -2$

$x_{02} = 1$

$x_{03} = 2$

Function:

$$f(x) = 0{,}5x^3 - 0{,}5x^2 - 2x + 2$$

Antiderivative:

$$F(x) = 0{,}5 \cdot \frac{1}{4}x^4 - 0{,}5\frac{1}{3}x^3 - 2\frac{1}{2}x^2 + 2x = \frac{1}{8}x^4 - \frac{1}{6}x^3 - x^2 + 2x$$

Area

$$\mathcal{F} = \left|\int_{-2,5}^{-2} f(x)\,dx\right| + \int_{-2}^{1} f(x)\,dx + \left|\int_{1}^{2} f(x)\,dx\right| + \int_{2}^{2,5} f(x)\,dx$$

$$\begin{aligned}\mathcal{F} &= |F(-2) - F(-2{,}5)| + F(1) - F(-2) + |F(2) - F(1)| + F(2{,}5) - F(2) = \\ &= |-4{,}67 + 3{,}76| + 0{,}96 - (-4{,}66) + |0{,}67 - 0{,}96| + 1{,}03 - 0{,}67 = \\ &= 0{,}90 + 5{,}625 + 0{,}29 + 0{,}36 = 7{,}175\end{aligned}$$

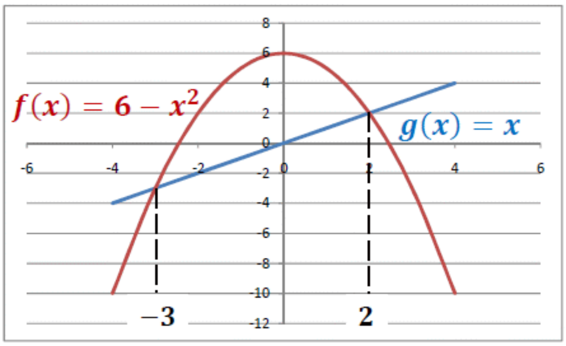
Area between Two Curves [one floor, one ceiling]

$$\begin{bmatrix}Area\ between \\ curves\end{bmatrix} = \int_a^b \left[\underbrace{\underbrace{f(x)}_{upper} - \underbrace{g(x)}_{lower}}_{one\ ceiling\ -\ one\ floor}\right] dx$$

Examples:

1. Compute the area of the region between the graphs of $y = x$ and $y = 6 - x^2$.

   To identify the top $y = f(x)$ and the bottom $y = g(x)$ and the interval $[a, b]$ we need a sketch.



Intersections:

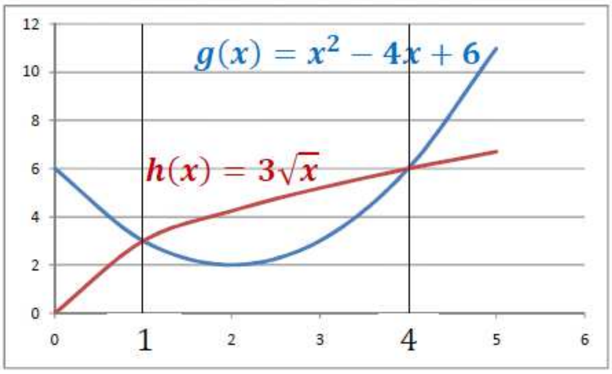
$$6 - x^2 = x$$
$$x^2 + x - 6 = 0$$
$$(x + 3)(x - 2) = 0$$
$$x = -3, 2$$
$$[a, b] = [-3,2]$$

Area: $$\int_{-3}^{2}((6 - x^2) - x)\,dx = \int_{-3}^{2}(6 - x^2 - x)\,dx = 6x\Big|_{-3}^{2} - \frac{1}{3}x^3\Big|_{-3}^{2} - \frac{1}{2}x^2\Big|_{-3}^{2}$$

$$= 6(2 - (-3)) - \frac{1}{3}(8 - (-27)) - \frac{1}{2}(4 - 9) = \frac{125}{6}$$

2. Compute the area of the region between two graphs:

$$g(x) = x^2 - 4x + 6 \text{ and } h(x) = 3\sqrt{x}$$



Intersections:

$$x^2 - 4x + 6 = 3\sqrt{x}$$

$$x_1 = 1,\ x_2 = 4$$

Area:
$$\int_1^4 \left(3\sqrt{x} - x^2 + 4x - 6\right) dx = 2x^{\frac{3}{2}} - \frac{1}{3}x^3 + 2x^2 - 6x \Big|_1^4 =$$

$$= \left(16 - \frac{64}{3} + 32 - 24\right) - \left(2 - \frac{1}{3} + 2 - 6\right) = \frac{8}{3} + \frac{7}{3} = 5$$

Sources:

Irina Kuzyakova: Computer Science and Mathematics (study course MES), summer semester 2014, part "Basics of Calculus"
Richard Delaware (Univ. of Missouri): Lectures (youtube.com)
Gregory L. Naber (Drexel University): Lectures (youtube.com)
Wikipedia