Derivatives of higher order

Notation for Derivatives of Derivatives [Higher order Derivatives]

1st Derivative:

$$f'(x),\ \frac{d}{dx}[f(x)],\ y',\ \frac{d}{dx}[y] = \frac{dy}{dx}, Df, D_x y$$

2nd Derivative:

$$f''(x),\ \frac{d}{dx}\left[\frac{d}{dx}f(x)\right] = \frac{d^2}{dx^2}[f(x)],\ y'',\ \frac{d}{dx}\left[\frac{d}{dx}(y)\right] = \frac{d^2y}{dx^2},\ D^2 f, D_x^2 y$$

The second derivative of y wrt x

For higher derivatives

$$f^{(n)}(x),\ \frac{d^n y}{dx^n} = \frac{d^n}{dx^n}[f(x)], D^n f, D_x^n y$$

The differentiations rules are the same

Exercise:

$$f(x) = 3x^4 - 2x^3 + x^2 - 4x + 2$$

$$f'(x) = 12x^3 - 6x^2 + 2x - 4$$

$$f''(x) = 36x^2 - 12x + 2$$

$$f^{(3)}(x) = 72x - 12$$

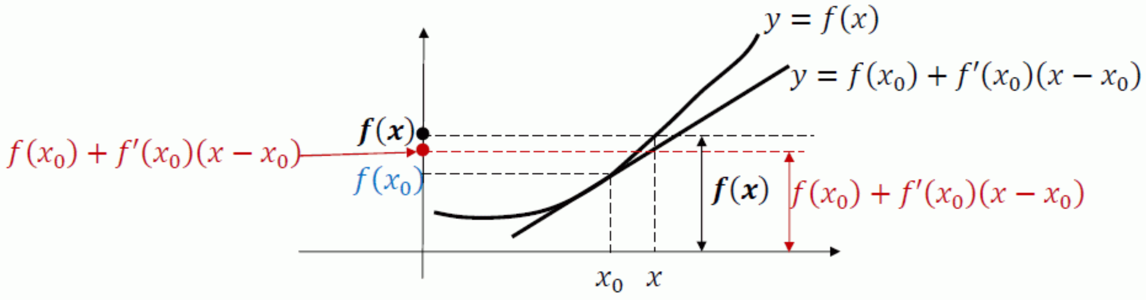
$$f^{(4)}(x) = 72$$

$$f^{(n)}(x) = 0\ for\ all\ n = 5,6,7 \ldots$$

Applications of derivatives

Derivative Application

Local Linear Approximations of Non-Linear Functions



Tangent line at $(x_0, f(x_0))$

$$y - f(x_0) = f'(x_0)(x - x_0)$$

$$y = f(x_0) + f'(x_0)(x - x_0)$$

For value of x near x_0 then

$$f(x) \approx f(x_0) + \underbrace{f'(x_0)(x - x_0)}_{tangent\ at\ x_0}$$

A local linear approximation of $f(x)$ near x_0

Another way of writing this:

Let $x - x_0 = \Delta x$, so $x = x_0 + \Delta x$

$$f(x_0 + \Delta x) \approx f(x_0) + f'(x_0)\Delta x$$

Better approximation: by including higher-order derivatives (but then nonlinear – the graph of *f* in the neighbourhood of x_0 is approximated by a polynomial curve):

Taylor's formula
(with $\Delta x = x - x_0$)

$$f(x_0 + \Delta x) = f(x_0) + \frac{f'(x_0)}{1!} \cdot \Delta x + \frac{f''(x_0)}{2!} \cdot (\Delta x)^2 + \frac{f'''(x_0)}{3!} \cdot (\Delta x)^3 + \ldots$$

Applications of derivatives:

Finding limits using differentiation

The rule of de l'Hospital

Limits of Quotients That Appear to be "Indeterminate": $\frac{0}{0}, \frac{\infty}{\infty}, 0 \cdot \infty, \infty - \infty$

1. Assumption: Suppose $\lim_{x \to a} \left[\frac{f(x)}{g(x)}\right]$

has the $\frac{0}{0}$ form

meaning both:

$$\lim_{x \to a} f(x) = 0$$

and

$$\lim_{x \to a} g(x) = 0$$

2. Assumption: Suppose f, g are both differentiable at a,

so f, g are both continuous at a

meaning

$$\left.\begin{aligned} \lim_{x \to a} f(x) = f(a) \\ \lim_{x \to a} g(x) = g(a) \end{aligned}\right\} f(a) = g(a) = 0$$

Observe:

$$\frac{f(x)}{g(x)} = \frac{f(x) - \overbrace{f(a)}^{=0}}{g(x) - \underbrace{g(a)}_{=0}} = \frac{\left[\frac{f(x)-f(a)}{x-a}\right]}{\left[\frac{g(x)-g(a)}{x-a}\right]}$$

$x \to a$ $\qquad$ $x \to a$

So,

$$\lim_{x\to a}\frac{f(x)}{g(x)} = \lim_{x\to a}\frac{\left[\frac{f(x)-f(a)}{x-a}\right]}{\left[\frac{g(x)-g(a)}{x-a}\right]} = \lim_{x\to a}\left[\frac{f'(x)}{g'(x)}\right]$$

Theorem:
The Rule of de l'Hospital (for the 0/0 situation)

If f, g are both differentiable on $I, a \in I$ and both

$$\lim_{x\to a} f(x) = 0 \text{ and } \lim_{x\to a} g(x) = 0$$

Then

$$\lim_{x\to a}\left[\frac{f(x)}{g(x)}\right] = \lim_{x\to a}\left[\frac{f'(x)}{g'(x)}\right]$$

A limit we hope exists and we hope it is easier to calculate

Note: The rule of de l'Hospital applies also to the ∞ / ∞ situation.

<u>Warning:</u> The calculation here is quite different from the application of the quotient rule for determining the derivative of *f/g* !

Examples:

$$\underbrace{\lim_{x\to 0}\left[\frac{sinx}{x}\right]}_{"\frac{0}{0}"} = \lim_{x\to 0}\left[\frac{cosx}{1}\right] = \cos(0) = 1$$

$$\underbrace{\lim_{x\to \pi/2}\left[\frac{1-sinx}{cosx}\right]}_{"\frac{0}{0}"} = \lim_{x\to \pi/2}\left[\frac{-cosx}{-sinx}\right] = \frac{0}{-1} = 0$$

$$\underbrace{\lim_{x\to \infty}\left[\frac{x^2}{e^x}\right]}_{"\frac{\infty}{\infty}"} = \lim_{x\to \infty}\left[\frac{2x}{e^x}\right] = \lim_{x\to \infty}\left[\frac{2}{e^x}\right] = 0$$

Finding other „indeterminate“ limits:

- Also apply to $\infty \cdot 0, \infty - \infty, 1^{\infty}, 0^0, \infty^0$

We have to reduce any indeterminate form to either $\frac{0}{0}$ and $\frac{\infty}{\infty}$

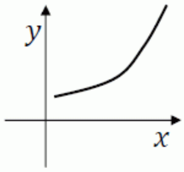
Example:

$$\underbrace{\lim_{x\to 0}[x \cdot lnx]}_{"0\cdot(-\infty)"} = \underbrace{\lim_{x\to 0}\left[\frac{lnx}{\frac{1}{x}}\right]}_{"\frac{-\infty}{\infty}"} = \lim_{x\to 0}\left[\frac{\frac{1}{x}}{-x^{-2}}\right] = \lim_{x\to 0}\left[-\frac{x^2}{x}\right] = -\lim_{x\to 0} x = 0$$

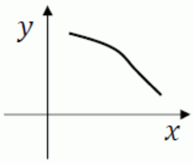
The Derivative Applied Analyzing the Graphs of Functions

Increasing and Decreasing Functions

Definition (Algebraic): A function f is increasing on same interval I, if for any x_1, x_2 in I $x_1 < x_2$ imply $f(x_1) < f(x_2)$



A function f is decreasing on same interval I, if for any x_1, x_2 in I $x_1 < x_2$ imply $f(x_1) > f(x_2)$



Constant function: not increasing, not decreasing

- f is increasing on an interval ⟺ Graph is rising from left to right
- f is decreasing on an interval ⟺ Graph is falling from left to right

Theorem: If f is continuous on $[a, b]$ and differentiable on (a, b)

Then

$$f'(x) > 0, all\ x \in (a, b) \quad \Rightarrow \quad f\ increasing\ on\ [a, b]$$

$$f'(x) < 0, all\ x \in (a, b) \quad \Rightarrow \quad f\ decreasing\ on\ [a, b]$$

$$f'(x) = 0, all\ x \in (a, b) \quad \Rightarrow \quad f\ constant\ on\ [a, b]$$

Local Maximums and Minimums

- f changes from increasing to decreasing at a relative (or local) maximum point
- f changes from decreasing to increasing at a relative (or local) minimum point

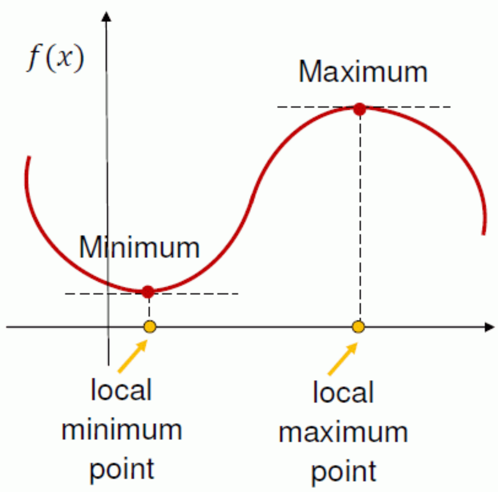
Definition. A function $y = f(x)$ has a local maximum at "c" (some point) (in some interval I) if for all x in I $f(x) \leq f(c)$.

Called a local maximum value for f

Definition. A function $y = f(x)$ has a local minimum at "c" (some point) (in some interval I) if for all x in I $f(x) \geq f(c)$.

Called a local minimum value for f

„Local extremum“ means either local maximum or local minimum.



Local =Relative

Definition:
An x_0 in the domain of f is a **critical point** for f if

$$f'(x_0) = 0 \quad \text{or} \quad f'(x_0) \text{ does not exist.}$$

Theorem:
Let f be defined on an open interval I which contains x_0.
If f has a local extremum at x_0, then x_0 must be a critical point of f.

But: The contrary is not necessarily true! Critical points are not automatically points of local max or min.
They are **candidates** for local max / min.

So remember:
Extrema occur at critical points, but not every critical point is the point of an extremum.

To determine the extrema we must do two things:

1. Find the critical points (compute $f'(x)$ and find out where it is either 0 or undefined)

2. "Test" each critical point to determine if it a relative maximum, a relative minimum, or neither

For the second, there are two "tests" available: The first derivative test and the second derivative test.

The 1st derivative Test for local Maximums and Minimums

Observe: [f continuous at critical point x_0]

- Local maximum $f' > 0\ -\ f' < 0$ The derivative changes sign from + to –

- Local minimum $f' < 0\ -\ f' > 0$ The derivative changes sign from – to +

Using these observations we have the 1st derivative test for local extrema

The 2nd Derivative Test for local Maximums and Minimums

- An alternative to the 1st derivative test. Use only if the 2nd derivative is easy to calculate
- Nice, because instead of looking to the left and right of x_0, you just look directly at x_0

Observe: Assume $f''(x_0)$ exists. [Thus, $f'(x_0)$ must exist]

So,

$$\begin{bmatrix} f'(x_0) = 0 \\ and\ f''(x_0) > 0 \end{bmatrix} \Rightarrow \begin{bmatrix} f\ has\ a\ local \\ minimum\ at\ x_0 \end{bmatrix}$$

$$\begin{bmatrix} f'(x_0) = 0 \\ and\ f''(x_0) < 0 \end{bmatrix} \Rightarrow \begin{bmatrix} f\ has\ a\ local \\ maximum\ at\ x_0 \end{bmatrix}$$

$$\begin{bmatrix} f'(x_0) = 0 \\ and\ f''(x_0) = 0 \end{bmatrix} \Rightarrow [Inconclusive]$$

Example: Find all local extrema of the function:

$$f(x) = -2x^3 + 3x^2 + 12x + 10$$

Solution:

$$f'(x) = -6x^2 + 6x + 12$$

$$-6x^2 + 6x + 12 = 0$$

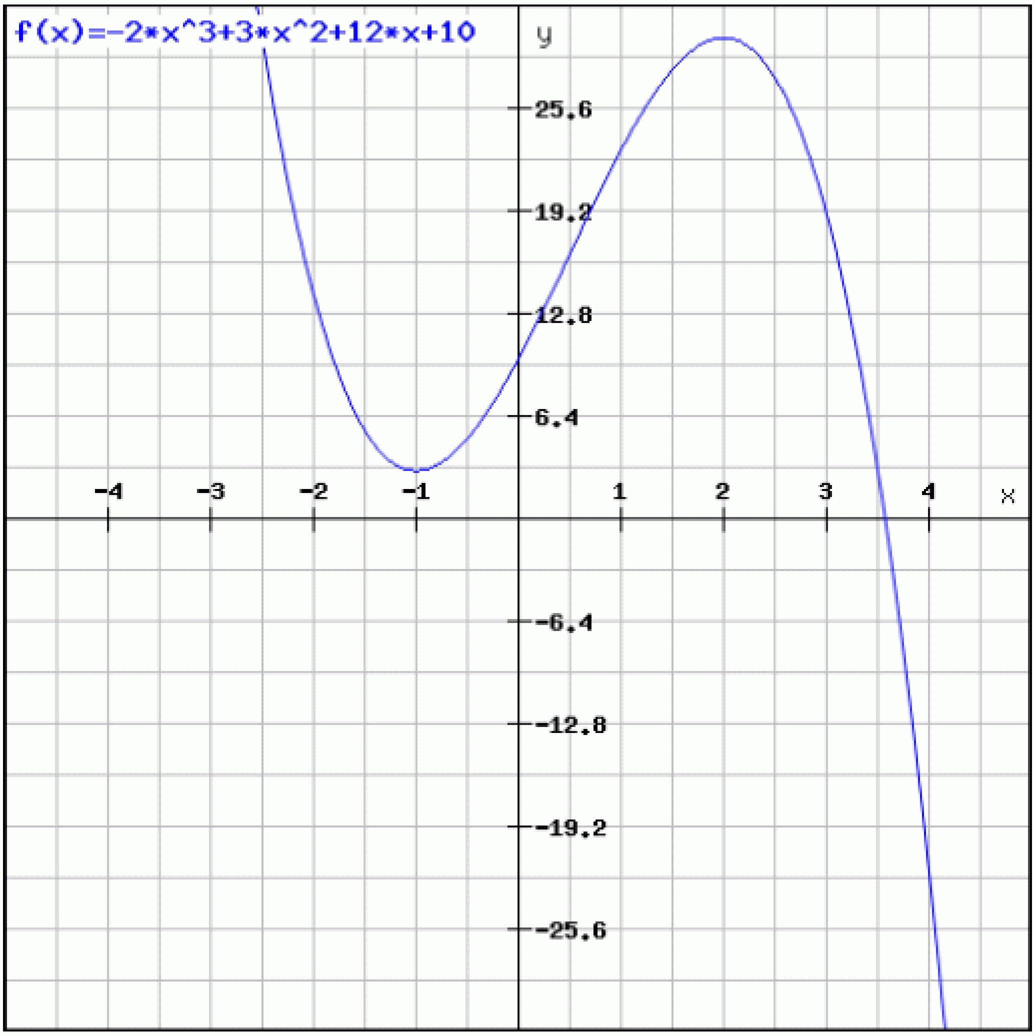
$$x_{1,2} = \frac{-6 \pm 18}{-12}$$

$$x_1 = 2, x_2 = -1$$

$$f''(x) = -12x + 6$$

$$x_1 = 2: \quad f''(x) = -12 \cdot 2 + 6 = -18 < 0: local\ maximum$$

$$x_2 = -1: \quad f''(x) = -12 \cdot (-1) + 6 = 18 > 0: local\ minimum$$



Example: $f(x) = |x^2 - x|$

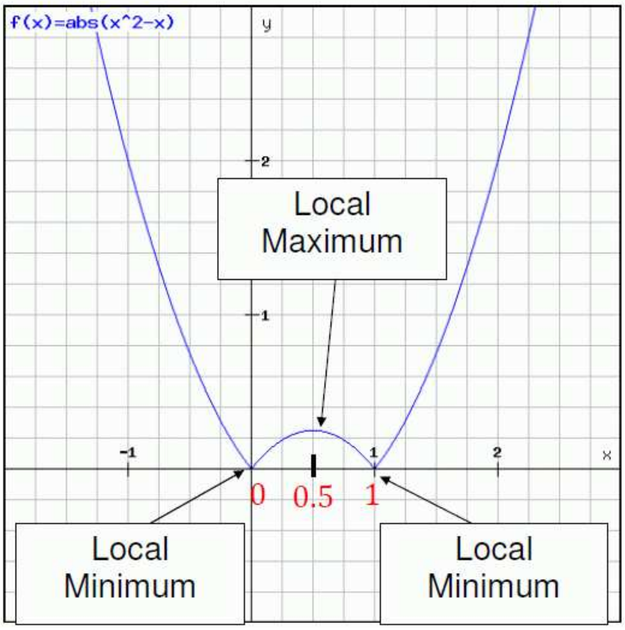
$$f'(x) = \begin{cases} 2x - 1 \; if \; x < 0 \\ -2x + 1 \; if \; 0 < x < 1 \\ 2x - 1 \; if \; x > 1 \end{cases}$$

Critical points:

$at\ x = 0.5 \quad f'(x) = 0$

$at\ x = 0 \quad f'(x)\ does\ not\ exist$

$at\ x = 1 \quad f'(x)\ does\ not\ exist$



Global (Absolute) Maximums and Minimums

Consider: the function $f(x)$, I is same Interval in the Domain of f and $x_0 \in I$

Definition:

- f has a global maximum at x_0 if $f(x_0) \geq f(x)$ at $x \in I$
- f has a global minimum at x_0 if $f(x_0) \leq f(x)$ at $x \in I$

We say "global extremum" for either

Global extrema on (finite) closed Intervals

Extreme Value Theorem

If

f is continuous
on closed I $[a, b]$, } both hypothesis necessary

then f has both a global maximum and global minimum [guaranteed!] – "Existence Theorem"

Further Theorem: Suppose f has a global extremum on an Interval (a, b) open. Then that extremum must occur at a critical point.

Summary:

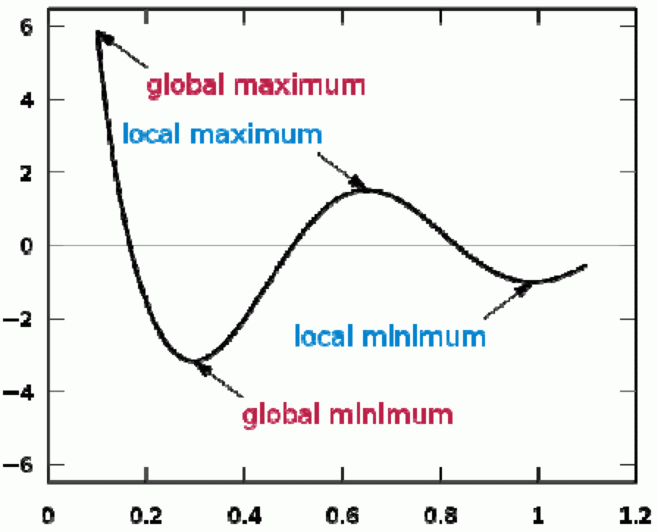
f continuous on $[a, b]$ $\Rightarrow$

1. f has both global extrema (min and max)
2. These occur either at a or b (endpoints) or where $f\,'(x) = 0$ or where $f\,'$ doesn't exist.

Finding global extrema:

1. Find all the critical points of f $[a, b]$
2. Evaluate f at these points, and at a and b
3. Largest value=global maximum

   Smallest value=global minimum



http://en.wikipedia.org/wiki/File:Extrema_example_original.svg

Applied Maximum and Minimum problem: Optimization Problem

“Optimization” (find the best”)

A strategy

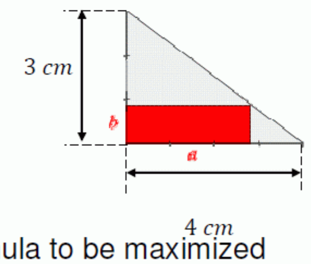
- Draw a sketch +label relevant quantities
- Find a formula for the one quantity to be maximized or minimized
- Use given information to write that formula as a function of **one** variable
- Find the domain of that variable
- Use the derivative to find the desired global max/min

Example: What is the biggest rectangle you can put inside a given triangle?

Given a right triangle of altitude $3\ cm$ an base $4cm$

Find a dimension of the rectangle of maximum area that can be inscribed in this triangle with one side along the base.

- A sketch

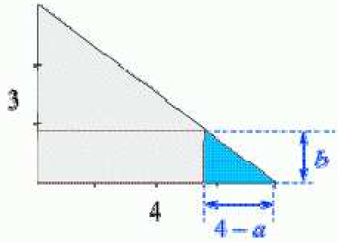


- A formula to be maximized

$$f = a \cdot b$$

We seek the maximum to the product $a \cdot b$ .We need to find a so that f is maximized

- The formula as a function of one variable



$$\frac{3}{4} = \frac{b}{4-a}$$

$$b = \frac{3(4-a)}{4}$$

$$f = a \cdot b = \frac{3a(4-a)}{4} = 3a - 0.75a^2$$

- Domain of a: $0 < a < 4$

- The derivative used

$$f'(a) = 3 - 1.5a = 0$$

$$a = 2$$

maximum or minimum?

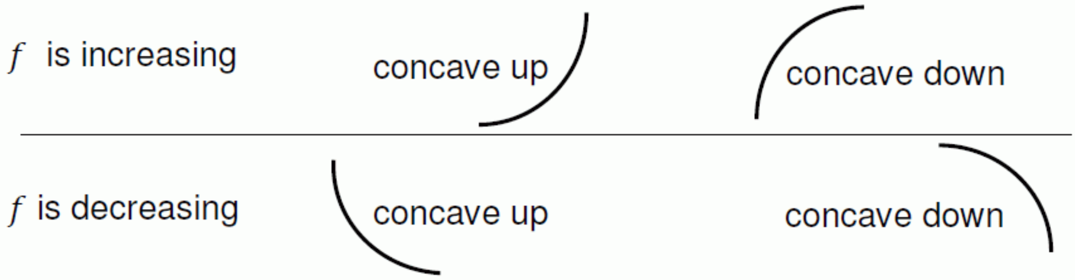
$f'' = -1.5 < 0$ - maximum

$$b = \frac{3(4-a)}{4} = 1.5$$

The 2nd Derivative Applied:

Function Concave Up and Concave Down

f can increase (or decrease) in two different way: concave up and concave down



A point at which f changes from concave up to concave down or from concave down to concave up is called an **inflection** point.

The 2nd Derivative Test

Definition: Let f have a derivative on open interval I

- f **concave up** on I means f' is **increasing** on I
- f **concave down** on I means f' is **decreasing** on I

To tell if a function (later f') is increasing/decreasing, we check its first derivative of (f'):

$$(f')' = f''$$

Theorem:

Suppose f is twice differentiable on I

$$\begin{cases} f''(x) > 0 \\ all\ x \in I \end{cases} \Rightarrow f \text{ is concave up in } I$$

$$\begin{cases} f''(x) < 0 \\ all\ x \in I \end{cases} \Rightarrow f \text{ is concave down in } I$$

When Concavity Changes: Inflection Points

Definition:

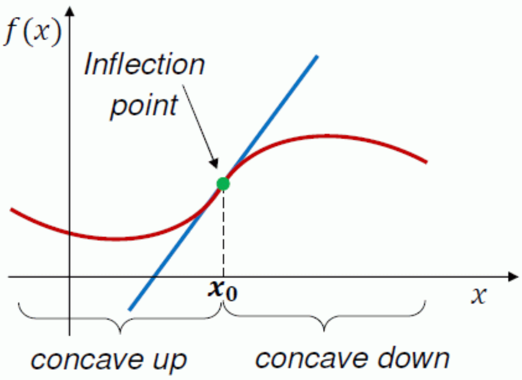
If f is continuous on open I and concavity changes at $(x_0, f(x_0))$

then we say: f has an **inflection point** at x_0 and $(x_0, f(x_0))$ is that inflection point.

$f''(x_0) = 0$ gives candidates for inflection points, but no guaranties:

$f'''(x_0) \neq 0$: inflection point

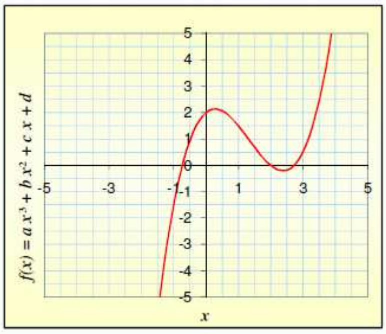
$f'''(x_0) = 0$: undefined



Examples:

function	**1.derivative**	**2. derivative**	**Concave up/down?**
$f(x) = x^2$	$2x$	$2 > 0$	concave up
$f(x) = -x^2$	$-2x$	$-2 < 0$	concave down
$f(x) = (e^{2x} + 4e^{-x})^2$	$4e^{4x} + 8e^{x} - 32e^{-2x}$	$16e^{4x} + 8e^{x} + 64e^{-2x} > 0$	concave up

Example: $f(x) = 0{,}5x^3 - 2x^2 + x + 2$



$$f'(x) = 1{,}5x^2 - 4x + 1;\ f''(x) = 3x - 4;\ f'''(x) = 3 \neq 0$$

$$3x - 4 < 0:\ x < \frac{4}{3};\qquad \left]-\infty; \frac{4}{3}\right] : concave\ down$$

$$3x - 4 > 0:\ x > \frac{4}{3};\qquad \left[\frac{4}{3};\ +\infty\right[: concave\ up$$

$$x = \frac{4}{3} : inflection\ point$$

Issues to investigate in a function diagram

with algebra:

- domain and range
- intersections with the x axis
- intersection with the y axis
- possibly symmetry

With Limits:

- Asymptotes
- End Behavior $x \to -\infty, x \to \infty$

With derivatives:

- Increasing/decreasing
- Local Extrema
- Concave up/down
- Inflection Points

Functions of two variables

A function of two variables x and y is a rule which assigns

to each ordered pair (x, y) of real numbers in some subset of the xy-plane, called the Domain of the function,

exactly one real number

$$z = f(x, y)$$

called the value of f at (x, y).

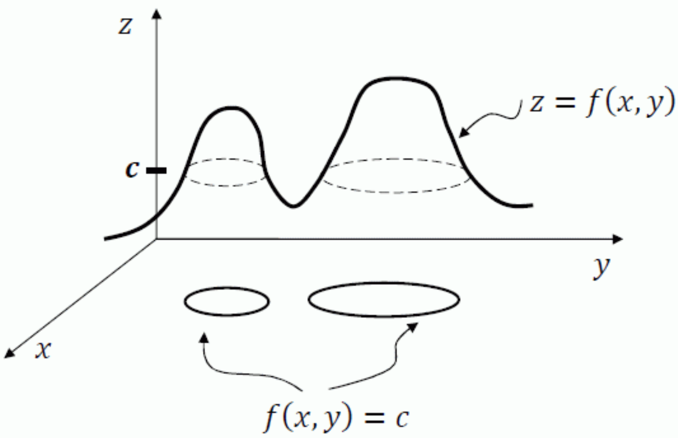
The value of f depends on two different parameters

Example: The temperature at the certain point on the surface of the earth $f(x, y)$, where x and y are longitude and latitude.

The graph of f

The graph of f is a surface in space. So for each value of x and y we have x, y in the (x, y) –plane, then we plot the point in space at position x, y: $z = f(x, y)$

It is possible to obtain something like a "picture" of a function $z = f(x, y)$ without drawing its graph in space. It is the **contour** plot. The graph is sliced by horizontal planes. It is a representing the function of two variables by the map.

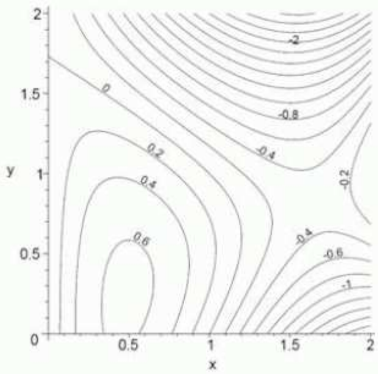


There are a bunch of curves. A **level** curve for $z = f(x, y)$ is a curve in the x, y - plane on which the function takes only one value, i.e. with an equation of the form

$$f(x, y) = c$$

for constant c

Draw enough of these, label each with the c it came from (so that you know how height it should be lifted to get to the graph) and you have some idea what the surface looks like.



Limits and continuity for function of two variables.

Recall:

$$\lim_{x \to x_0} f(x) = L$$

If $f(x)$ can be made as close as we like to L by choosing x sufficiently close (but not equal) to x_0

$\lim_{x \to x_0} f(x) = L$ exists if and only if both

$$\lim_{x \to x_0^-} f(x) = L$$

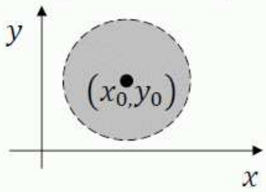
and

$$\lim_{x \to x_0^+} f(x) = L$$

are equal

For $f(x, y)$ the definition looks essentially the same:

Given $f(x, y)$ an a point (x_0, y_0) in the plane with f defined at least "near" (x_0, y_0)

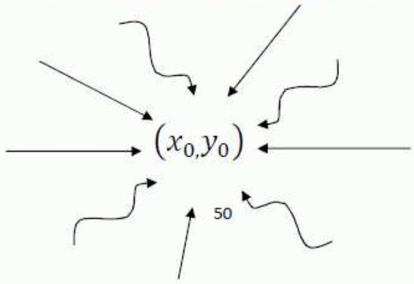


We say that

$$\lim_{(x,y)\to(x_0,y_0)} f(x, y) = L$$

if $f(x, y)$ can be made as close as we like to L choosing (x, y) sufficiently close (but not equal) to (x_0, y_0).

This time, however, instead of just two there are infinitely many "approaches" to $(x_{0,}, y_{0,})$ and, in order for the limit to exist, they must all give the same result.



Continuity

Recall: $f(x)$ is continuous at x_0 if $\lim_{x\to x_0} f(x) = f(x_0)$.

Implicit in this is

- x_0 is in the domain of $f(x)$ so $f(x_0)$ exists
- $\lim_{x\to x_0} f(x)$ exists
- these two are the same

For function of two variables the definition is the same

$f(x, y)$ is continuous at (x_0, y_0) if

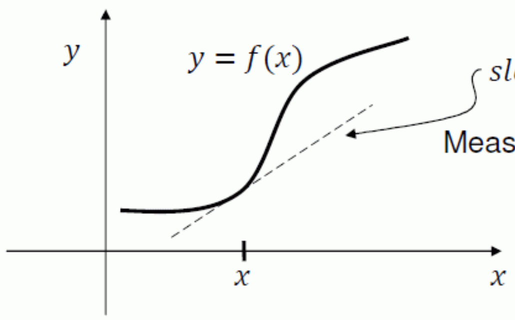
$$\lim_{(x,y)\to(x_0,y_0)} f(x, y) = f(x_0, y_0)$$

If this is true for every (x_0, y_0) in the domain of $f(x, y)$ we say simply that $f(x, y)$ is continuous

- Polynomials are continuous everywhere
- Rational functions are continuous wherever the denominator is nonzero
- Sums, differences and products of continuous functions are continuous
- Quotients of continuous functions are continuous wherever the denominator is nonzero
- If $f(x,y)$ is continuous and $g(u)$ is a continuous function of one variable, then $g(f(x,y))$ is continuous

Partial Derivatives

Recall: Given $y = f(x)$ and x in its Domain



$$slope = f'(x) = \frac{dy}{dx} = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

Measure of the rate at which f is changing at x

Now suppose $y = f(x,y)$ and (x,y) is a point in its domain.

"Rate at which f is changed at (x,y)" makes no sense since f can change at different rate in different directions at (x,y)

Partial Derivatives: Rates of changes in the x-direction and in the y-direction

Slope of a tangent line in x-direction = **partial derivative** of f with respect to x

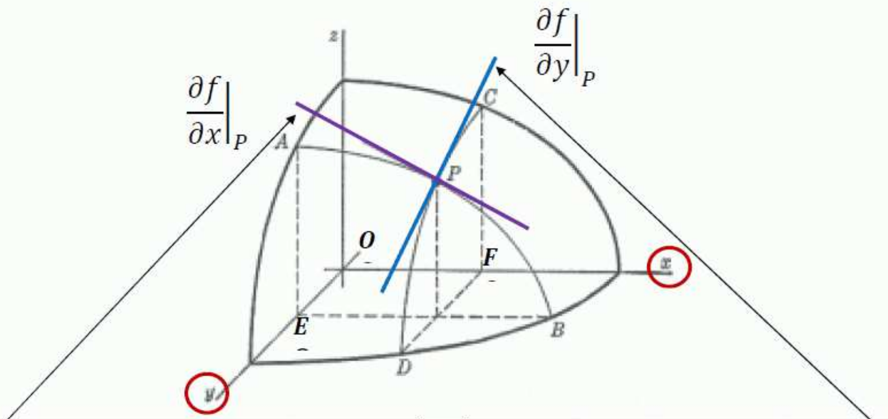
$$= \frac{\partial f}{\partial x} = \lim_{h \to 0} \frac{f(x+h, y) - f(x, y)}{h}$$

- hold y fixed and differentiate with respect to x as usual.

Slope of a tangent line in y-direction = partial derivative of f with respect to y

$$= \frac{\partial f}{\partial y} = \lim_{h \to 0} \frac{f(x, y+h) - f(x, y)}{h}$$

- hold y fixed and differentiate with respect to y with usual.



Partial derivative of f with respect to x

$$= \frac{\partial f}{\partial x} = \lim_{h \to 0} \frac{f(x+h, y) - f(x, y)}{h}$$

- hold y fixed and differentiate with respect to x as usual.

Partial derivative of f with respect to y

$$= \frac{\partial f}{\partial y} = \lim_{h \to 0} \frac{f(x, y+h) - f(x, y)}{h}$$

- hold x fixed and differentiate with respect to y as usual.

Short-hand Notation for Partial Derivatives

If $z = f(x, y)$, we can write the partial derivative functions as

$$\frac{\partial f}{\partial x} = \frac{\partial z}{\partial x} = \frac{\partial}{\partial x} f(x, y) = f_x = f_x(x, y) = D_x f = D_1 f = \cdots$$

$$\frac{\partial f}{\partial y} = \frac{\partial z}{\partial y} = \frac{\partial}{\partial y} f(x, y) = f_y = f_x(x, y) = D_y f = D_2 f = \cdots$$

We can define the partial derivatives at a point (a, b) as

$$f_x(x, y) = \left.\frac{\partial f}{\partial x}\right|_{(a,b)}$$

$$f_y(x, y) = \left.\frac{\partial f}{\partial y}\right|_{(a,b)}$$

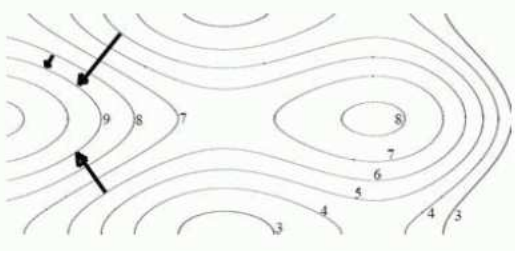
Examples:

$$f(x, y) = x \cdot siny, \qquad \frac{\partial f}{\partial x} = siny, \qquad \frac{\partial f}{\partial y} = x \cdot cosy$$

$$f(x, y) = x^2 + y^2, \qquad \frac{\partial f}{\partial x} = 2x, \qquad \frac{\partial f}{\partial y} = 2y$$

Gradient

The **gradient** of a function f points in the direction of the greatest rate of increase of the function, and whose magnitude is that rate of increase.



The **gradient** of f:

$$\nabla f = grad\ f = \begin{pmatrix} \frac{\partial f}{\partial x} \\ \frac{\partial f}{\partial y} \end{pmatrix}$$

The **gradient** of f at the point (x_0, y_0):

$$\nabla f(x_0, y_0) = \begin{pmatrix} \frac{\partial f}{\partial x}(x_0, y_0) \\ \frac{\partial f}{\partial y}(x_0, y_0) \end{pmatrix}$$

Tangent plane

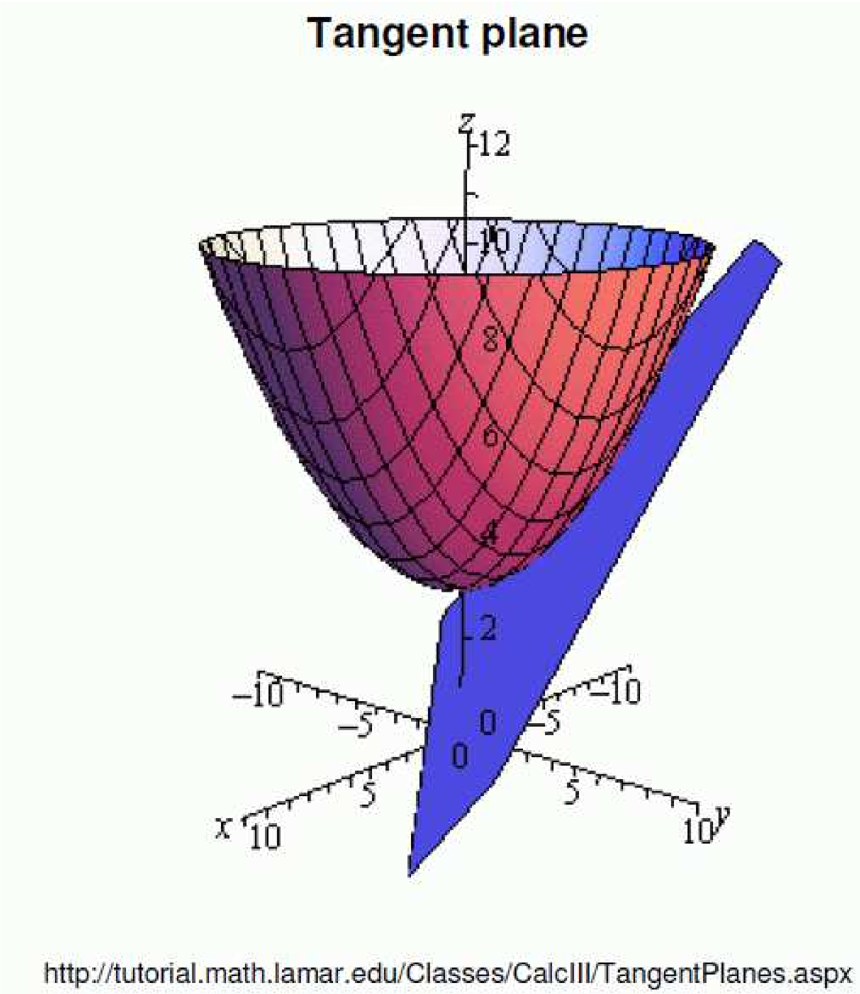
Let (x_0, y_0) be any point of a surface function $z = f(x, y)$ Then the surface has a nonvertical tangent plane at (x_0, y_0) with equation

$$T_{(x_0,y_0)} = f(x_0, y_0) + \begin{pmatrix} \frac{\partial f}{\partial x}(x_0, y_0) \\ \frac{\partial f}{\partial y}(x_0, y_0) \end{pmatrix} \cdot \begin{pmatrix} x - x_0 \\ y - y_0 \end{pmatrix}$$

$$= f(x_0, y_0) + \underbrace{\nabla f(x_0, y_0)}_{\substack{Gradient\ at\ point \\ (x_0,y_0)}} \begin{pmatrix} x - x_0 \\ y - y_0 \end{pmatrix}$$

A tangent plane to a function $f(x_0, y_0)$ at the point (x_0, y_0) is a plane that just touches the graph of the function at the point $\big((x_0, y_0), f(x_0, y_0)\big)$.

Approximation formula = the graph is close to its tangent plane.



Example: Find the equation of a tangent plane to:

$$f(x, y) = x^2 + y^2$$

At the point $(x_0, y_0) = (1,2)$

Solution:

$$\nabla f(x_0, y_0) = (2x \quad 2y)(1,2) = \begin{pmatrix} 2 \\ 4 \end{pmatrix}$$

$$T(x, y) = f(1,2) + \nabla f(1,2)\begin{pmatrix} x-1 \\ y-2 \end{pmatrix} = 5 + (2 \quad 4)\begin{pmatrix} x-1 \\ y-2 \end{pmatrix}$$

$$= 5 + 2(x-1) + 4(y-2) = -5 + 2x + 4y$$

Second order partial derivatives: $f(x,y)$

$$\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2 f}{\partial x^2} = f_{xx}$$

$$\frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial^2 f}{\partial y^2} = f_{yy}$$

$$\left.\begin{aligned}\frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2 f}{\partial y \partial x} = f_{yx} \\ \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial^2 f}{\partial x \partial y} = f_{xy}\end{aligned}\right\} \textit{mixed second order partial derivatives}$$

Note: If the two mixed second order partial derivatives are continuous then they will be equal.

$$\boxed{\frac{\partial^2 f}{\partial y \partial x} = \frac{\partial^2 f}{\partial x \partial y}} \qquad \boxed{f_{xy} = f_{yx}}$$

So, the order of taking partial derivatives of a function $f(x,y)$ can be interchanged

Examples:

$$f(x,y) = x^3 y - x^2 y^2$$

$$f_x = 3x^2 y - 2xy^2, \; f_y = x^3 - 2x^2 y$$

$$f_{xx} = 6xy - 2y^2, \; f_{yy} = -2x^2$$

$$f_{xy} = 3x^2 - 4xy, \; f_{yx} = 3x^2 - 4xy$$

Sources:

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