3. Continuity and derivatives

Continuous functions

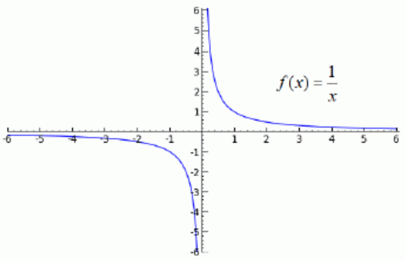
Continuity of functions at a single point $x = c$

Definition: A function f is continuous at $x = c$ provided all three conditions are satisfied.

1. $f(x)$ is defined [f exists at $x = c$]
2. $\lim_{x \to c} f(x)$ exists [equals a real number]
3. $\lim_{x \to c} f(x) = f(c)$

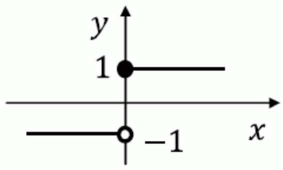
If not, f is discontinuous at $x = c$

Some examples of discontinuous functions



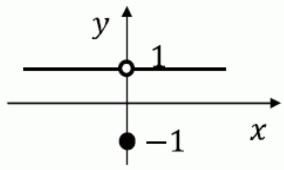
$$f(x) = \frac{1}{x}$$

$f(0)$ not defined



$$f(x) = \begin{cases} 1, x \geq 0 \\ -1, x < 0 \end{cases}$$

$f(0)$ defined, but $\lim_{x \to 0} f(x)$ does not exist



$$f(x) = \begin{cases} 1, x \neq 0 \\ -1, x = 0 \end{cases}$$

$f(0)$ defined, and $\lim_{x \to 0} f(x)$ exists, but they are not the same

Intuitively, $f(x)$ is continuous at $x = a$ if the graph of f does not break at $x = a$ and does not have a "hole" (i.e., undefined function value) at this point.

If f is not continuous at $x = a$ (i.e., if the graph of f does break or has a "hole" there), then $x = a$ is a *discontinuity* of f.

Note: If $x = a$ is an endpoint for the Domain of $f(x)$, then $\lim_{x \to a} f(x)$ in the definition is replaced by the appropriate one-sided limit, e.g.

$$f(x) = \sqrt{x}$$

Is defined on $[0, \infty)$ and is continuous at $x = 0$ because $f(0) = 0$ and

$$\lim_{x \to 0^+} f(x) = 0$$

Continuity of a function on an interval

f is continuous on some open interval (*a*, *b*) or (–∞, +∞) if *f* is continuous at each *x* = *c* in the interval.
(Two-sided limits are possible here at each *c* in the interval.)

What about f defined on some closed interval $[a, b]$: No two-sided limits at a or b?

Definition: at $x = c$ f is continuous

from the left,

if $\lim_{x \to c^-} f(x) = f(c)$

from the right,

if $\lim_{x \to c^+} f(x) = f(c)$

Definition: f is continuous on $[a, b]$, if

1. f is continuous on (a, b)
2. f is continuous "from the right" at a
3. f is continuous "from the left" at b

Properties and combinations of continuous functions

Recall: If $p(x)$ is a polynomial function, then $\lim_{x \to c} p(x) = p(c)$

So, every polynomial function is continuous everywhere.

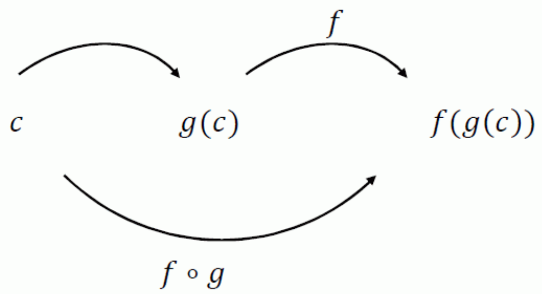
Suppose, f, g are continuous at $x = c$

Theorem: $f + g$; $f - g$; $f \cdot g$ are all also continuous at $x = c$
$\frac{f}{g}$ is continuous at $x = c$ provided $g(c) \neq 0$
[otherwise, $\frac{f}{g}$ is discontinuous at $x = c$]

So, every rational function is continuous at every point where the bottom is not zero.

The composition of continuous functions is also continuous

Theorem: If g is continuous at c and f is continuous at $g(c)$ then $f \circ g$ is continuous at c

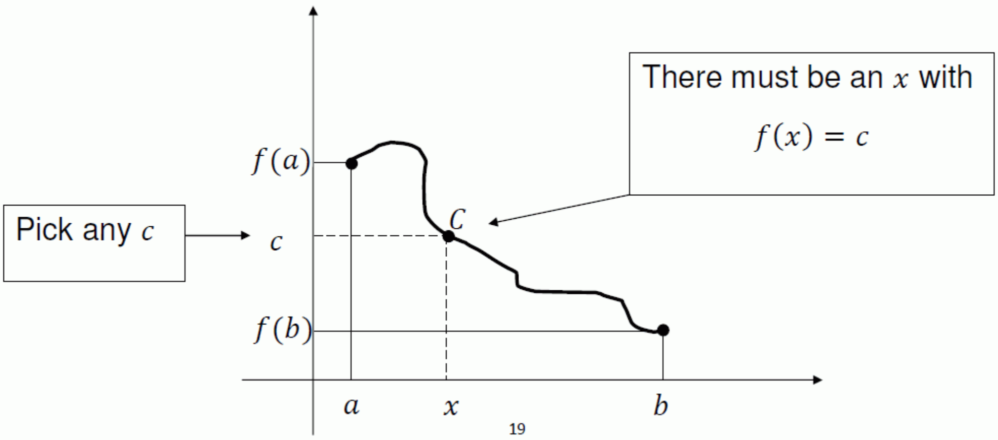


Continuity of Functions: Applications

The Intermediate Value Theorem and Approximating Roots $f(x) = 0$

Intermediate Value Theorem

If f is continuous on $[a, b]$ and c is between $f(a)$ and $f(b)$, or equal to one of them, then there is at least one value of x in $[a, b]$ such that $f(x) = c$



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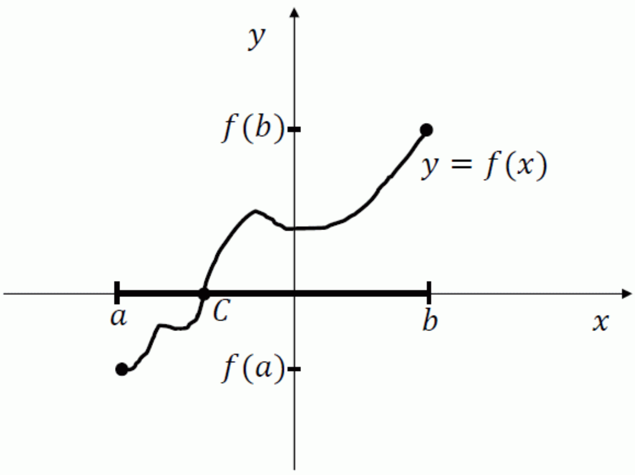
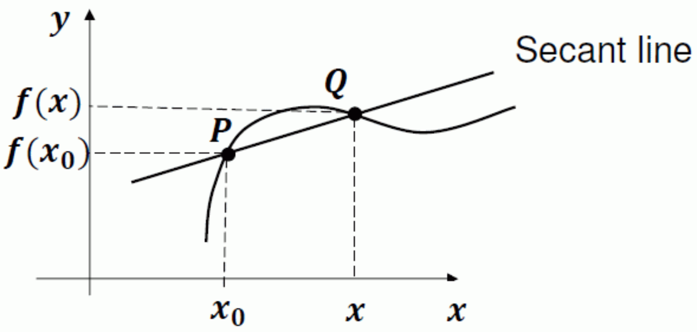
Approximating Roots $f(x) = 0$

Theorem: If f is continuous on $[a, b]$ and $f(a), f(b)$ are non zero with opposite signs, then there is at least one "solution" of $f(x) = 0$ in (a, b)



The Derivative of a Function

Measuring Rates of Change of a function $f(x)$

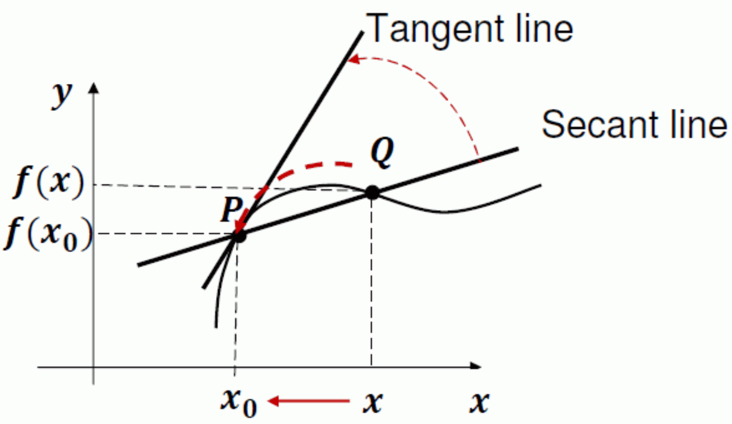


Average rate of change of y with respect to x over $[x_0, x]$

$$= r_{average} = \frac{\text{"}change\ in\ y\text{"}}{\text{"}change\ in\ x\text{"}} = \frac{f(x) - f(x_0)}{x - x_0}$$

- Slope of secant line through the points $x_0, f(x_0)$ and $x, f(x)$

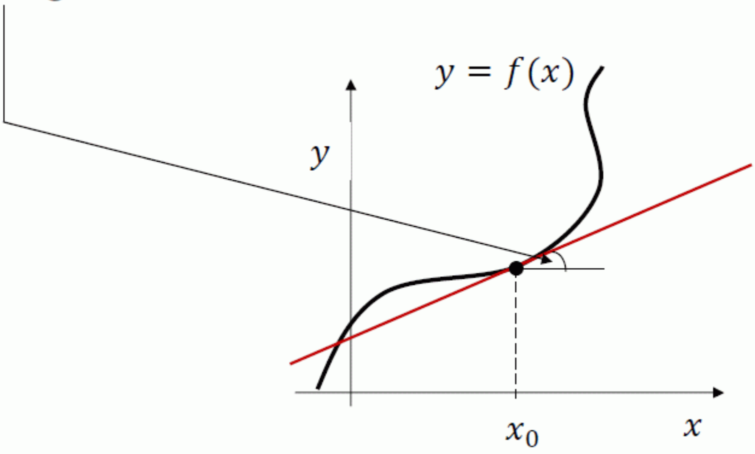
Instantaneous rate of change of y with respect to x at point x_0



$$r_{instantaneous} = \lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0}$$

- Slope of tangent line at $x_0, f(x_0)$ [provided the limit exists]

Slope of Tangent Lines



Definition:

$$[Tangent\ Slope\ at\ x_0] = \lim_{x \to x_0} [Secant\ slope\ between\ x_0\ and\ x]$$

So,

$$m_{Tangent} = \lim_{x \to x_0} \left[\frac{f(x) - f(x_0)}{x - x_0} \right]$$

[provided the limit exists]

Since the tangent line passes through $(x_0, f(x_0))$, its equation is

$$y - f(x_0) = m_{tangent}(x - x_0)$$

Alternate notation:

$$x = x_0 + h; \quad m_{Tangent} = \lim_{h \to 0} \left[\frac{f(x_0 + h) - f(x_0)}{h} \right]$$

What is a Derivative

Definition: The function f' [f prime of x] derived from f and defined by

$$f'(x) = \lim_{h \to 0} \left[\frac{f(x+h) - f(x)}{h} \right]$$

is called the derivative of f with respect to (wrt) x

The process of finding a derivative is called **differentiation.**

Example:

Find $f'(x)$, if $f(x) = x^2 + 1$

Solution:

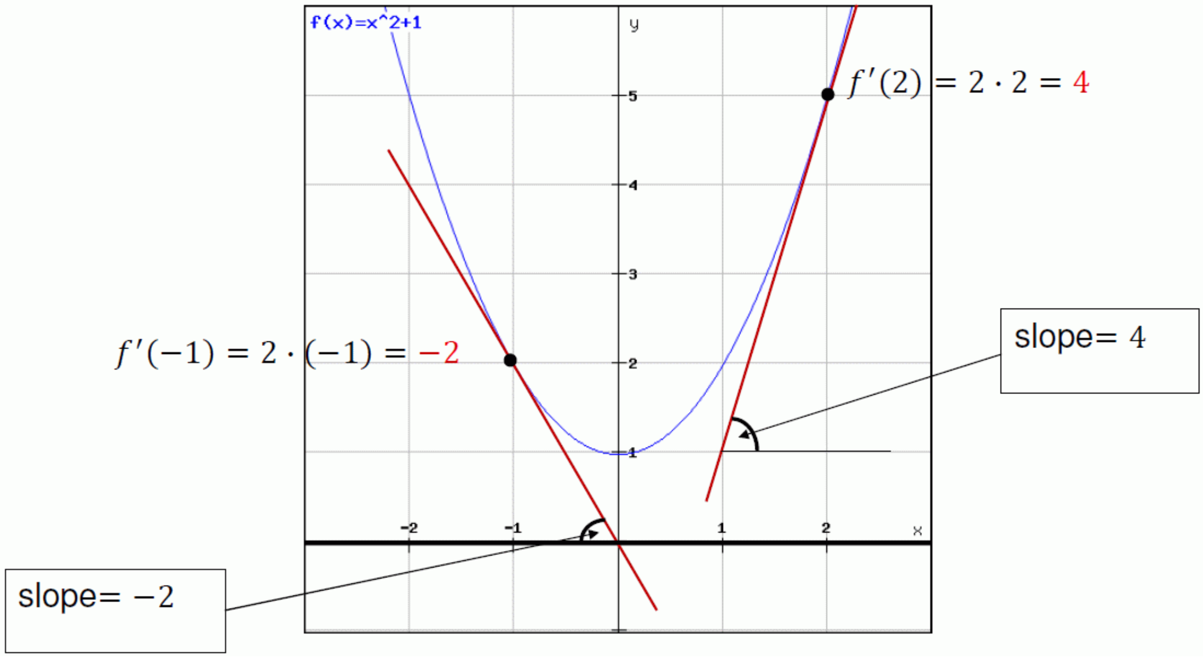
$$f'(x) = \lim_{h \to 0} \left[\frac{f(x+h) - f(x)}{h} \right] = \lim_{h \to 0} \left[\frac{(x+h)^2 + 1 - (x^2 + 1)}{h} \right] =$$

$$\lim_{h \to 0} \left[\frac{1}{h} (\cancel{x^2} + 2hx + h^2 + \cancel{1} - \cancel{x^2} - \cancel{1}) \right] = \lim_{h \to 0} \left[\frac{2hx + h^2}{h} \right] = 2x$$

The **derivative function** $f'(x)$ tells us the value of the derivative for any point on the original function.

When we evaluate the derivative function for a given x value, we get a number which is the derivative at a point (i.e., the rate of change of f, or the slope of the graph of f)

Function: $f(x) = x^2 + 1$ Derivative function: $f'(x) = 2x$



Other example: Let us check that the tangent slope of *f*(*x*) = *mx* + *b* is *m* everywhere.

Solution:

$$f'(x) = \lim_{h \to 0} \left[\frac{f(x+h) - f(x)}{h} \right] = \lim_{h \to 0} \left[\frac{1}{h} \big(m(x+h) + b - (mx + b) \big) \right] =$$

$$\lim_{h \to 0} \left[\frac{1}{h} (mx + mh + b - mx \quad b) \right] = \lim_{h \to 0} \left[\frac{1}{h} mh \right] = \lim_{h \to 0} m = m$$

Notation for differentiation

There is no single uniform notation for differentiation. Instead, several different notations for the derivative of a function or variable have been proposed by different mathematicians. The usefulness of each notation varies with the context, and it is sometimes advantageous to use more than one notation in a given context.

Lagrange's notation: The notation $\boldsymbol{f'(x)}$

One of the most common modern notations for differentiation is due to Italian mathematician **Joseph Louis Lagrange** (1736-1813) and uses the **prime mark.**

Euler's notation is due to Swiss mathematician Leonhard Euler (1707-1883)

Euler's notation uses a differential operator, denoted as D, which is prefixed to the function so that the derivatives of a function f are denoted by

$$Df$$

When taking the derivative of a dependent variable $y = f(x)$ it is common to add the independent variable x as a subscript to the D notation, leading to the alternative notation

$$D_x y$$

Leibnitz Notation:

It is particularly common when the equation $y = f(x)$ is regarded as a functional relationship between dependent and independent variables y and x. In this case the derivative can be written as:

$$\frac{dy}{dx} \quad or \quad \frac{d}{dx}[y] \quad or \quad \frac{d}{dx}(y)$$

This notation was previously introduced by the German mathematician Baron Wilhem Gottfried von Leibniz (1646-1716).

Since $y = f(x)$, we can also write

$$\frac{df}{dx} \quad or \quad \frac{d(f(x))}{dx} \quad or \quad \frac{d}{dx}[f(x)]$$

This is also called **differential notation**, where dy and dx are **differentials**.

With Leibniz's notation, the value of the derivative of y at a point $x = x_0$ can be written as:

$$f'(x_0) = \frac{d}{dx}[f(x)]\bigg|_{x = x_0} = \frac{dy}{dx}\bigg|_{x = x_0}$$

Functions: Differentiable (or not!) at a single point?

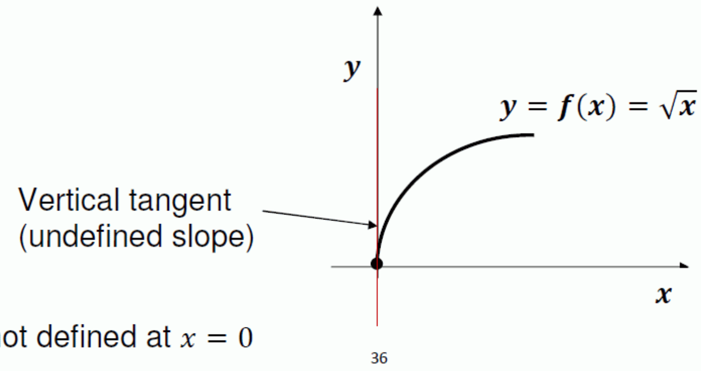
We say: f is differentiable at x_0 [has a derivative at x_0] if $f'(x_0)$ exists.

The process of finding derivatives of function is called **differentiation**

If a function has a derivative at a point it is said to be **differentiable** at that point.

e.g. $f(x) = \sqrt{x}$ is differentiable at every point in its domain except $x = 0$

Geometric reason:



$f'(x) = \dfrac{1}{2\sqrt{x}}$ is not defined at $x = 0$

A function differentiable at a point is continuous at that point

Theorem: If f is differentiable at x_0 then f is continuous at x_0

Proof: Since f is differentiable at x_0 we know

$$f'(x_0) = \lim_{h \to 0} \left[\frac{f(x_0 + h) - f(x_0)}{h} \right]$$

exists.

To show f is continuous at x_0 we must show [definition of a continuous function]

$$\lim_{x \to x_0} f(x) = f(x_0)$$

We can rewrite:

$$\lim_{x \to x_0} [f(x) - f(x_0)] = 0$$

Rewriting once more, we need to show with $x = x_0 + h$

$$\lim_{h\to 0}[f(x_0+h) - f(x_0)] = 0$$

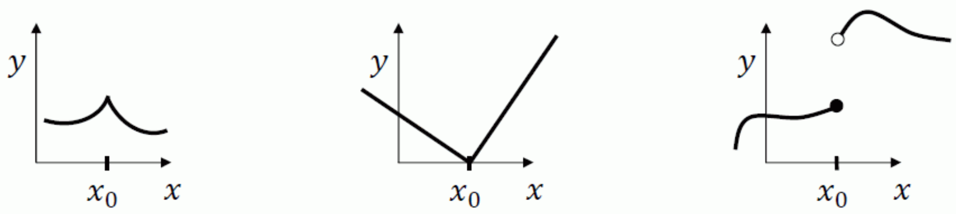
$$\lim_{h\to 0}[f(x_0+h) - f(x_0)] = \lim_{h\to 0}\left[(f(x_0+h) - f(x_0)) \cdot \underbrace{\frac{h}{h}}_{=1}\right]$$

$$= \lim_{h\to 0}\left[\frac{f(x_0+h) - f(x_0)}{h} \cdot h\right] = \underbrace{\lim_{h\to 0}\left[\frac{f(x_0+h) - f(x_0)}{h}\right]}_{f'(x_0)} \cdot \lim_{h\to 0} h$$

$$= f'(x_0) \cdot 0 = 0$$

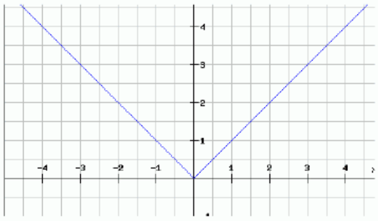
f can fail to be differentiable!

Here are the ways in which $f(x)$ can fail to be differentiable at x_0



Graphically: Graphs of differentiable functions are "smooth" in that they do not have "sharp points."

Differentiability implies continuity, but continuity doesn't imply differentiability.



Example: $f(x) = |x|$

The function $f(x) = |x|$ is continuous

But

The function $f(x) = |x|$ is not differentiable at $x = 0$

$$\lim_{h \to 0} \frac{f(0+h) - f(0)}{h} = \lim_{h \to 0} \frac{|h|}{h}$$

which does not exist because

$$\frac{|h|}{h} = \begin{cases} 1, h > 0 \\ -1, h < 0 \end{cases}$$

The function $f(x) = |x|$ is **continuous** at 0 but is not **differentiable** at 0.

Functions Differentiable on an Interval

- On open intervals: a function must be differentiable at each point of the interval (must have 2-sided limit at each point)
- On interval with endpoints: a function must be differentiable at each point on the open interval (2-sided limit) and have a left/right hand limits at the end points

Definition:

Left Hand Derivative

$$f_-{'}(x) = \lim_{h \to 0^-} \left[\frac{f(x+h) - f(x)}{h}\right]$$

Right Hand Derivative

$$f_+{'}(x) = \lim_{h \to 0^+} \left[\frac{f(x+h) - f(x)}{h}\right]$$

Finding Derivatives

1. Differentiation technique:

$$f'(x) = \lim_{h \to 0} \left[\frac{f(x+h) - f(x)}{h} \right]$$

2. The derivative of any constant function is zero

$$(c)' = 0$$

Obvious: Horizontal line has a horizontal tangent at each point

3. The Power Rule:

For any real number n

$$(x^n)' = nx^{n-1}$$

Examples:

function	**1.derivative**
$f(x) = cx = cx^1$	$f'(x) = c \cdot 1 \cdot x^{1-1} = c$
$f(x) = x^2$	$f'(x) = 2 \cdot x^{2-1} = 2x$
$f(x) = x^3$	$f'(x) = 3 \cdot x^{3-1} = 3x^2$

Multiplying by a Constant; Sum and Difference Rules

Theorem: If f, g are differentiable at x and c is any real number
Then

$$\left(cf(x)\right)' = cf'(x)$$

$$(f(x) + g(x))' = f'(x) + g'(x)$$

$$\left(f(x) - g(x)\right)' = f'(x) - g'(x)$$

Exercise: Find $f'(x)$

function	1.derivative
$f(x) = 2 + x^{0,5}$	$f'(x) = 0 + 0{,}5x^{0,5-1} = 0{,}5x^{-0,5}$
$f(x) = 5x^2 - 3x$	$f'(x) = 2 \cdot 5 \cdot x^{2-1} - 3 \cdot x^{1-1} = 10x - 3$
$f(x) = \frac{6}{\sqrt{x^3}} = 6x^{-\frac{3}{2}}$	$f'(x) = 6\left(-\frac{3}{2}\right)x^{-\frac{3}{2}-1} = -\frac{18}{2}x^{-\frac{5}{2}} = -\frac{9}{\sqrt{x^5}}$

The Product Rule

Observe:

$$\big(f(x)\cdot g(x)\big)' \neq f'(x)\cdot g'(x)$$

Example:

$$f(x) = 1, \qquad g(x) = x$$

$$f'(x) = 0,\ g'(x) = 1,$$

$$f'(x)\cdot g'(x) = 0\cdot 1 = \boxed{0}$$

$$f(x)\cdot g(x) = 1\cdot x = x$$

$$\big(f(x)\cdot g(x)\big)' = x' = \boxed{1}$$

So,

$$\big(f(x)\cdot g(x)\big)' = 1 \neq f'(x)\cdot g'(x) = 0$$

Theorem: If

f, g are differentiable at x

then

$$\big(f(x)\cdot g(x)\big)' = f'(x)g(x) + f(x)g'(x)$$

We can write too using Leibniz Notation

$$\frac{d}{dx}[uv] = u\frac{dv}{dx} + v\frac{du}{dx}$$

Generalized Product Rule:

$$\left(\prod_{i=1}^{n} f_i\right)' = (f_1\cdot f_2\cdot f_3\cdot \dots \cdot f_n)'$$

$$= f_1'\cdot f_2 \dots f_n + f_1\cdot f_2'\cdot \dots \cdot f_n + \cdots + f_1\cdot f_2 \dots f_{n-1}' f_n + f_1\cdot f_{n-1}\cdot f_n'$$

Example:

$$f(x) = 2x^3(x-1)$$

Solution:

$$f'(x) = 3 \cdot 2x^2(x-1) + 2x^3 \cdot 1 = 6x^3 - 6x^2 + 2x^3 = 8x^3 - 6x^2$$

or:

$$f(x) = 2x^3(x-1) = 2x^4 - 2x^3$$

$$f'(x) = 2 \cdot 4x^3 - 2 \cdot 3x^2 = 8x^3 - 6x^2$$

The Quotient Rule

Observe:

$$\left(\frac{f(x)}{g(x)}\right)' \neq \frac{f'(x)}{g'(x)}$$

Example:

$$f(x) = 1, \qquad g(x) = x, \qquad \frac{f(x)}{g(x)} = \frac{1}{x},$$

$$\boxed{\left(\frac{f(x)}{g(x)}\right)'} = (x^{-1})' = \left(\frac{1}{x}\right)' (-1) \cdot x^{-1-1} = \boxed{-\frac{1}{x^2}}$$

$$f'(x) = 0, \qquad g'(x) = 1, \qquad \boxed{\frac{f'(x)}{g'(x)}} = \frac{0}{1} = \boxed{0}$$

$$\left(\frac{f(x)}{g(x)}\right)' = -\frac{1}{x^2} \neq \frac{f'(x)}{g'(x)} = 0$$

Theorem: If f, g are differentiable at x, Then

$$\left(\frac{f(x)}{g(x)}\right)' = \frac{f'(x)g(x) - f(x)g'(x)}{g(x)^2}$$

We also write:

$$\frac{d}{dx}\left[\frac{u}{v}\right] = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$$

Special case:

$$\left(\frac{1}{f(x)}\right)' = \frac{0 \cdot f(x) - 1 \cdot f'(x)}{f(x)^2} = -\frac{f'(x)}{f(x)^2}$$

Example:

$$f(x) = \frac{3x^2}{5 - x}$$

Solution

$$f'(x) = \frac{3 \cdot 2x(5 - x) - 3x^2(-1)}{(5 - x)^2} = \frac{30x - 6x^2 + 3x^2}{(5 - x)^2} = \frac{-3x^2 + 30x}{(5 - x)^2}$$

The Chain Rule: Derivatives of Composition of functions

Motivating example: $f(x) = (x^2 + 1)^{100}$. Find $f'(x)$

Our only technique is to multiply this out – very tedious.

Instead, think of $(x^2 + 1)^{100}$ as the composition of two functions.

Suppose

$$f(x) = x^{100}$$

$$g(x) = x^2 + 1$$

Then

$$f\big(g(x)\big) = f(x^2 + 1) = (x^2 + 1)^{100}$$

We can use the derivatives of x^{100} and $x^2 + 1$ to calculate the derivative $[wrt\ x]$ of

$$y = (x^2 + 1)^{100}$$

Rewrite

$$y = (x^2 + 1)^{100}$$

as

$$y = u^{100}, where\ u = x^2 + 1$$

Then

$$y'(u) = \frac{dy}{du} = 100u^{99}, and\ u'(x) = \frac{du}{dx} = 2x$$

To get $y'(x) = \frac{dy}{dx}$ we multiply

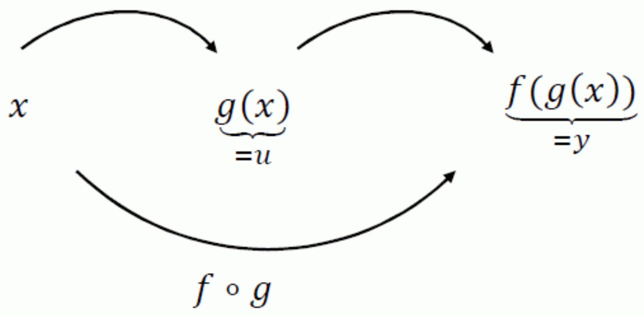
$$\frac{dy}{dx} = \underbrace{\frac{dy}{du}}_{outer} \cdot \underbrace{\frac{du}{dx}}_{inner}$$

$$y'(x) = y'(u) \cdot u'(x) = 100(x^2 + 1)^{99} \cdot 2x = 200x \cdot (x^2 + 1)^{99}$$

Theorem [The "Chain" Rule]

If g is differentiable at x and f is differentiable at $g(x) = u$

Then $y = (f \circ g)(x)$ is differentiable at x



and $(f \circ g)'(x) = f'(g(x)) \cdot g'(x)$.

Exercise: Find $f'(x)$:

$$f(x) = 4(x^2 - 1)^2$$

Solution:

$$f(x) = 4\underbrace{(x^2 - 1)}_{u}{}^2$$

$$y(u) = 4u^2; \quad u(x) = x^2 - 1$$

$$f'(x) = y'(u) \cdot u'(x) = 4 \cdot 2(x^2 - 1)2x = 16x^3 - 16x = 16x(x^2 - 1)$$

Derivatives of trigonometric functions

[sin *x*]' = cos *x*

$$[cosx]' = -sinx$$

$$[tanx]' = \frac{1}{cos^2x}$$

$$[cotx]' = -\frac{1}{sin^2x}$$

Exercises: Find $f'(x)$ (Chain Rule!)

function	1.derivative
$f(x) = cos2x$	$f'(x) = -sin2x \cdot 2 = -2sin2x$
$f(x) = sin\,(x^2)$	$f'(x) = cos(x^2) \cdot 2x = 2xcos(x^2)$
$f(x) = cos^2x$	$f'(x) = 2cosx(-sinx)$

Derivatives of Inverse Trigonometric Functions

$$[arcsinx]' = \frac{1}{\sqrt{1-x^2}}$$

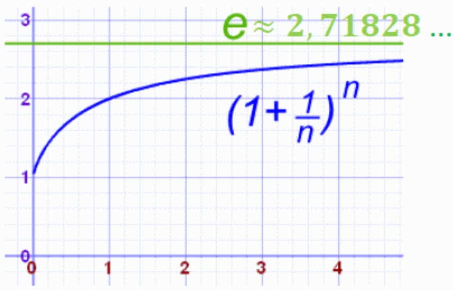
$$[arccosx]' = -\frac{1}{\sqrt{1-x^2}}$$

$$[arctanx]' = \frac{1}{1+x^2}$$

$$[arccotx]' = -\frac{1}{1+x^2}$$

The Natural Exponential Function e^x

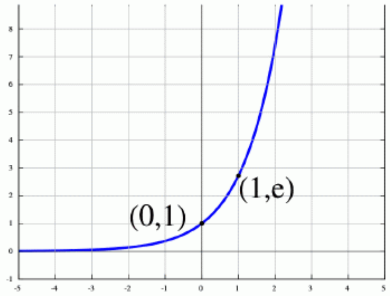
Definition: "e", Euler's number, is that number which approaches $\left(1+\frac{1}{n}\right)^n$, as $n \to \infty$



The number is called after Leonhard Euler, a Swiss mathematician

e is irrational, i.e. **cannot** be expressed as a ratio of integers.

A natural exponential function in standard form is $f(x) = e^x$



$Dom\ f = \mathbb{R}$, $Ran\ f = (0, \infty)$

- No x-intercepts, y-intercept $(0,1)$
- Horizontal asymptote $y = 0$
- f passes through $(1, e)$
- is increasing

Derivative of general exponential function:

$$\frac{d}{dx}[b^x] = (b^x)' = b^x lnb$$

Important case:

If $b = e$

$$(e^x)' = e^x lne = e^x$$

Natural Logarithm

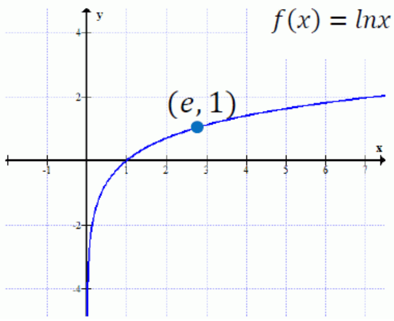
Definition: When $e^y = x$

Then base e logarithm of x is $ln(x) = log_e(x) = y$

Natural Logarithm Function:

$Dom\ f = (0, \infty)$, $Ran\ f = \mathbb{R}$

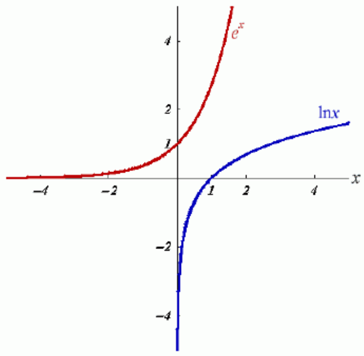
- x-intercept $(1,0)$, no y-intercepts,
- Vertical asymptote $x = 0$
- f passes through $(e, 1)$
- is increasing



The natural logarithm function $ln(x)$ is the inverse function $f^{-1}(x)$ of the exponential function $f(x) = e^x$

For $x > 0$, $f(f^{-1}(x)) = e^{ln(x)} = x$

Or $f^{-1}(f(x)) = ln(e^x) = x$



The graph of an inverse relation is the reflection of the original graph over the line $y = x$

Basic Logarithm Rules:

Product rule: $\boldsymbol{ln(xy) = ln(x) + ln(y)}$

Quotient rule: $\boldsymbol{ln\left(\frac{x}{y}\right) = ln(x) - ln(y)}$

Power rule: $\boldsymbol{ln(x^y) = y \cdot ln(x)}$

Change of base $\boldsymbol{\log_b x = \frac{lnx}{lnb}}$

$$(lnx)' = \frac{1}{x}, \quad for\ x > 0$$

$$(\log_b x)' = \frac{1}{lnb}\frac{1}{x}, \quad for\ x > 0$$

$$\log_b x = \frac{lnx}{lnb}$$

constant

Exercises: Find $f'(x)$ (Chain Rule!)

function	derivative
$f(x) = ln(2x-1)$	$f'(x) = \frac{1}{2x-1} \cdot 2 = \frac{1}{x-0{,}5}$
$f(x) = \frac{1}{lnx} = (lnx)^{-1}$	$f'(x) = -(lnx)^{-2} \cdot \frac{1}{x} = -\frac{1}{x(lnx)^2}$
$f(x) = xln(3-x^2)$	$f'(x) = ln(3-x^2) - \frac{2x^2}{(3-x^2)}$

Exercises:

function	1. derivative
$f(x) = e^{5x}$	$f'(x) = 5e^{5x}$
$f(x) = \frac{e^{5x}}{x^2} = e^{5x} \cdot x^{-2}$	$f'(x) = 5e^{5x} \cdot x^{-2} + e^{5x}(-2)x^{-3} = \frac{e^{5x}(5x-2)}{x^3}$
$f(x) = \sqrt{e^{2x} + x}$	$f'(x) = \frac{1}{2}(e^{2x} + x)^{-\frac{1}{2}} \cdot (2e^{2x} + 1)$
$f(x) = 2^x$	$f'(x) = (ln2)2^x$
$f(x) = 2^{3x}$	$f'(x) = 3(ln2)2^{3x}$
$f(x) = x \cdot 2^{3x}$	$f'(x) = 2^{3x} + 3x(ln2)2^{3x}$