Part III: Basics of Calculus

We return to *functions* with real numbers as values.

They are very often used for modelling dependencies or the behaviour of systems in physics and other sciences.

The main teaching objective of the course: To become acquainted with the basics of **calculus**, that includes the theories of differentiation, integration, measure, limits, infinite series, and analytic functions.

Topics for the next 4 weeks:

- Sequences, sums, series
- Limits of sequences and of functions
- Differentiation
- Curve discussion and extreme value problems
- Integration

1. Sequences

Consider the infinite „list" of terms:

$$\frac{1}{1}, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \dots$$

Formula for the *n*-th term: $1/n$

In general: $a_1, a_2, a_3, \dots, a_n, \dots$

We call this mathematical object a *sequence*
and can define it as a *function* from IN to IR:

$$a(1), a(2), a(3), \dots, a(n), \dots$$

Short notation: (a_n)

In practice, a sequence is an ordered list of real numbers that most often follows some rule (or pattern) to determine the next term in the list.

A sequence is often given by the *n*-th term formula (also called the *general term*).

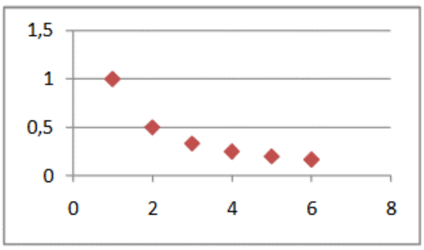
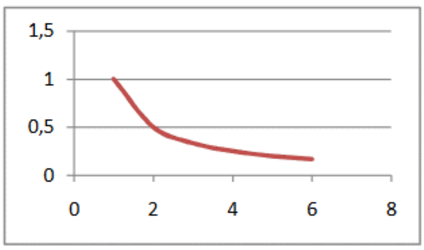
Exercise: Write the first 5 terms:

$$a_n = \frac{1}{2^n}, n = 1,2,3,4, \dots$$

Solution:

$$\frac{1}{2^1}, \frac{1}{2^2}, \frac{1}{2^3}, \frac{1}{2^4}, \frac{1}{2^5}$$

Distinction between a sequence and a function on IR (or on an interval of real numbers):

Sequence	**Function**
$a_n = \frac{1}{n}, n \in \mathbb{N}$	$f(x) = \frac{1}{x}, x \in [1, \infty)$
$Ran\ a = \left\{1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \dots\right\}$	$Ran\ f(x) = (0,1]$
	
Graph: Consisting of discrete discontinuous Dots	Graph: Continuous Curve

(„*Ran*" means „Range" here, i.e., the set of all obtained values.)

Warning: You cannot determine a sequence from only a finite number of terms!

Example:

a_1	a_2	a_3
1	3	9
1	3	9
1	3	9

(Some) possible solutions:

a_1	a_2	a_3	a_4	$\dots$	a_n
1	3	9	27	$\dots$	3^{n-1}
1	3	9	19	$\dots$	$1+2(n-1)^2$
1	3	9	11	$\dots$	$8n+\frac{12}{n}-19$

Recursion

Often a sequence is given by a recursive formula

- Stating its 1st term (s), then
- Writing a formula for the n^{th} term involving some preceding terms. This is called **a recursive formula**

Example:

$$a_1 = 1$$

$$\underbrace{a_n}_{subsequent} = 4 \cdot \underbrace{a_{n-1}}_{previous} \; : recursive\ formula$$

Solution:

$$a_1 = 1$$

$$a_2 = 4 \cdot a_1 = 4 \cdot 1 = 4$$

$$a_3 = 4 \cdot a_2 = 4 \cdot 4 = 16$$

$$a_4 = 4 \cdot a_3 = 4 \cdot 16 = 64$$

Limits of sequences

Some sequences "approach" a number as you move out of the sequence: e.g. the sequence

$$1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots$$

"approaches" 0

Definition:

A sequence (a_n) ***converges*** to a number L (called then the *limit* of the sequence) iff *any interval* around L (however small) contains ***nearly all*** the terms of the sequence,
which means: … all terms except finitely many,
which means: … all terms which come after some index n.

In this case we write:

$$L = \lim_{n \to \infty} a_n$$

If no such number exists we say that the sequence (a_n) ***diverges***.

Example 1: $(3^n) = (3; 3^2; 3^3; \ldots)$ diverges.

Example 2: $((1/2)^n) = (1/2; 1/4; 1/8; 1/16; \ldots)$ converges to 0.

Example 3: $(1^n) = (1; 1; 1; 1; \ldots)$ converges to 1.

For power sequences in general, (x^n) converges to 0 if $-1 < x < 1$, converges to 1 if $x = 1$, and diverges for every other value of x.

Shorthand: Summation notation.

The sum of n terms can be written as

$$a_1 + a_2 + a_3 + \cdots + a_n = \sum_{k=1}^{n} a_k$$

- Where k ends
- The form (formula) for the k^{th} term of our sequence
- „Sigma“- greek „s“ [stands for: sum]
- Index, starts at $k = 1$

Examples

$$\sum_{k=1}^{n} \left(\frac{1}{k}\right) = \frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n}$$

$$1^2 + 2^2 + 3^2 + \cdots + n^2 = \sum_{k=1}^{n} (k)^2$$

$$\sum_{k=5}^{8} [(-1)^{k+1} \cdot 2^k] = 2^5 - 2^6 + 2^7 - 2^8$$

Remark: The summation index must not necessarily be named *k*.
Other variable names can be used.
It must not necessarily begin at 1 and end at *n*.

Properties of sums

Let (a_n), (b_n) be sequences and c some real number. Then:

1. $\sum_{k=1}^{n}(c \cdot a_k) = c \cdot \sum_{k=1}^{n}(a_k)$

2+3. $\sum_{k=1}^{n}(a_k \pm b_k) = \sum_{k=1}^{n} a_k \pm \sum_{k=1}^{n} b_k$

4. $\sum_{k=1}^{n} a_k = \sum_{k=1}^{j} a_k + \sum_{k=j+1}^{n} a_k$, where $1 < j < n$ breaks into 2 pieces

5. $\sum_{k=1}^{n} c = n \cdot c$

Examples:

1.

$$\sum_{k=3}^{8} 9 = 9 + 9 + 9 + 9 + 9 + 9 = 6 \cdot 9$$

2.

$$\sum_{k=1}^{3} 5 \cdot \frac{1}{k} = 5\frac{1}{1} + 5\frac{1}{2} + 5\frac{1}{3} = 5\left(\frac{1}{1} + \frac{1}{2} + \frac{1}{3}\right)$$

Example:

Given are the following measurements for y_{ij}:

j \ i	1	2	3	4	5
1	1	2	1	3	6
2	3	5	3	1	5
3	4	3	2	5	1
4	6	8	2	3	2

Compute the following sums:

$$\sum_{i=3}^{5} \sum_{j=2}^{4} y_{ij}$$

Solution:

j \ i	**1**	**2**	**3**	**4**	**5**
1	1	2	1	3	6
2	3	5	3	1	5
3	4	3	2	5	1
4	6	8	2	3	2

$$\sum_{i=3}^{5}\sum_{j=2}^{4} y_{ij} = \sum_{i=3}^{5}(y_{i2}+y_{i3}+y_{i4}) = y_{32}+y_{33}+y_{34}+y_{42}+y_{43}+y_{44}+y_{52}+y_{53}+y_{54}$$
$$= 3+2+2+1+5+3+5+1+2 = 24$$

Partial sums

Given an infinite sequence (a_k), the sum of its first n terms is

$$a_1 + a_2 + a_3 + \ldots + a_n,$$

which we call its n-th *partial sum*.

(This is at the same time the n-th partial sum of the "infinite sum" $a_1 + a_2 + a_3 + \ldots$; we will come back to this concept.)

Notation:

$$S_n = \sum_{k=1}^{n} a_k, \quad n = 0,1,2, \ldots$$

A partial sum is a sum of part of the sequence

Example: "*Arithmetic sequence*"

Definition: Let $a, d \in$ IR. An arithmetic sequence has the standard form (a, $a+d$, $a+2d$, $a+3d$, …, $a+nd$).

Equivalent recursive definition:
$a_1 = a$ (first term), $a_{n+1} = a_n + d$ (for n = 1; 2; 3; …)

(i.e., d is the difference between any consecutive members of the sequence.)

The n-th partial sum of an arithmetic sequence:

Let (a_n) be an arithmetic sequence with first term *a* and common difference *d*. Then its *n*-th partial sum is:

$$S_n = \sum_{i=1}^{n}(a + (i-1)d) = \sum_{i=1}^{n} a + \sum_{i=1}^{n} d(i-1) =$$

$$na + d\sum_{i=1}^{n}(i-1) = na + d \cdot \frac{n(n-1)}{2} =$$

$$\frac{n}{2}[2a + (n-1)d] = \frac{n}{2}[a + a + (n-1)d]$$

$$\frac{n}{2}[a + a_n]$$

$$\sum_{i=1}^{n}(i-1) = \frac{n(n-1)}{2}$$

See below

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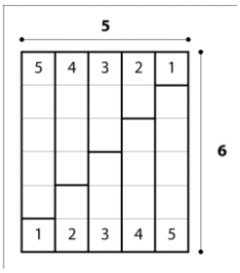
The sum of the first n natural numbers

Question: What is the sum of the first n natural numbers?

Answer:

$$S_n = \frac{n(n+1)}{2}$$

Geometric proof for $n = 5$



We see that we have a big rectangle with the its sides 5 and $5+1$. The rectangle has $2(1+2+3+4+5)$ squares inside. So $2(1+2+3+4+5) = 5(5+1)$ and $1+2+3+4+5 = \frac{5(5+1)}{2}$

http://www.9math.com/book/sum-first-n-natural-numbers

Example: "*Geometric sequence*"

Definition: Let a, $r \in$ IR where $r \neq 0$. A geometric sequence has the standard form

$(a, ar, ar^2, ar^3, \ldots)$.

r is called the *common ratio* of the sequence. (It is the ratio of any two consecutive members of the sequence.)
Equivalent recursive definition:
$a_1 = a$ (first term), $a_{n+1} = a_n \cdot r$ (for $n = 1; 2; 3; \ldots$).

The direct n-th term formula for a geometric sequence: $a_n = ar^{n-1}$

The n-th partial sum of a geometric sequence:

The n-th partial sum of the above geometric sequence is

$$S_n = a + ar + \cdots + ar^{n-1} = a\left(\frac{1-r^n}{1-r}\right)$$

($n = 1; 2; 3; \ldots$ and $r \neq 1$).

Proof: Assume $r \neq 1$.

$$S_n = a + ar + ar^2 + \cdots + ar^{n-2} + ar^{n-1}$$

$$-$$

$$rS_n = ar + ar^2 + \cdots + ar^{n-2} + ar^{n-1} + ar^n$$

$$S_n - rS_n = a - ar^n$$

$$S_n(1-r) = a(1-r^n)$$

$$\boxed{S_n = a\left(\frac{1-r^n}{1-r}\right)}$$

Observe:
The n-th partial sums S_n of a sequence $(a_k) = (a_1, a_2, a_3, \dots)$ form their own sequence (S_n):

$(S_n) = (a_1, a_1+a_2, a_1+a_2+a_3, \dots)$.

Series

The sum of the terms of a sequence is called a **series**.

Given an infinite sequence of numbers $\{ a_n \}$, a **series** is informally the result of adding all those terms together: $a_1 + a_2 + a_3 + \cdots$

These can be written more compactly using the summation symbol $\sum$. The **index of summation**, k takes consecutive integer values from the **lower limit**, 1 to the **upper limit**, n. The term a_k is a **general term**.

Finite series	**Infinite series**
$\sum_{k=1}^{n} a_k$	$\sum_{k=1}^{\infty} a_k$

A **finite series** is a summation of a finite number of terms. An **infinite series** has an infinite number of terms and an upper limit of infinity.

Convergence of infinite series

If the sequence $\{S_n\}$ of partial sums **converges** to some real number L i.e. a limit

$$\lim_{n\to\infty} S_n = L$$

exists,

then the series is said to converge to L.

In this case we can write:

$$\sum_{k=1}^{\infty} a_k = \lim_{n\to\infty} S_n = L$$

L is also called the **sum of the series**. In general, we say that an **infinite series** has a **sum** if the **partial sums** form a sequence that has a real **limit**.

If the limit of the sequence of partial sums *exists* and is finite,

then the *series* is called ***convergent***.

If the limit of the sequence of partial sums does *not exist* or is plus or minus infinity,

then the *series* is called ***divergent***.

Convergent and divergent infinite series

Example: Geometric series

Definition: The expression

$$a + ar + ar^2 + \cdots + ar^{k+1} + \cdots = \sum_{k=1}^{\infty} ar^{k-1}$$

with 1^{th} term a + common ratio $r \neq 0$ is called an **infinite geometric series**.

Theorem:

If $|r| < 1$, then the infinite geometric series $a + ar + ar^2 + \cdots + ar^{k+1} + \cdots$ has a finite sum for any constant a

$$\sum_{k=1}^{\infty} ar^{k-1} = \frac{a}{1-r}$$

Warning: If $|r| \geq 1$, then the $\frac{a}{1-r}$ sum formula is false.

Proof idea: Recall the n^{th} partial sum

$$S_n = a \cdot \left(\frac{1-r^n}{1-r}\right) = \frac{a}{1-r} - \frac{a \cdot r^n}{1-r}$$

Of course, as $n \to \infty$ $S_n \to$ Series "sum"

Since here $|r| < 1$, experience suggests that as $n \to \infty$, $|r|^n \to 0$ hence that as $n \to \infty$,

$$S_n = \frac{a}{1-r} - \underbrace{\frac{a \cdot r^n}{1-r}}_{\to 0} \to \frac{a}{1-r}$$

The geometric series **diverges** whenever $r \leq -1$ or $r \geq 1$:

Example: $r = 1; a = 1$

$$\sum_{k=0}^{\infty} (-1)^k = 1 - 1 + 1 - 1 + 1 \dots$$

diverges because its sequence of partial sums is

$$S_0 = 1$$

$$S_1 = 1 - 1 = 0$$

$$S_2 = 1 - 1 + 1 = 1$$

$$S_3 = 1 - 1 + 1 - 1 = 0$$

And the sequence $\{1,0,1,0, \dots\}$ diverges.

Infinite convergent series: Examples

Many so-called **elementary functions** can be defined by series.

The exponential function e^x may be defined by the following power series:

$$e^x = \sum_{k=0}^{\infty} \frac{x^k}{k!} = 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24} + \cdots, \quad x \in \mathbb{R}$$

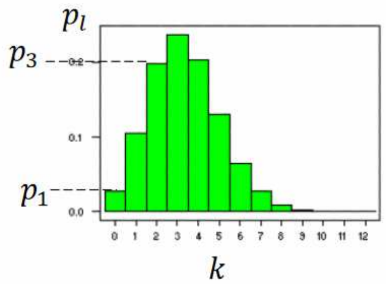
Cosine function

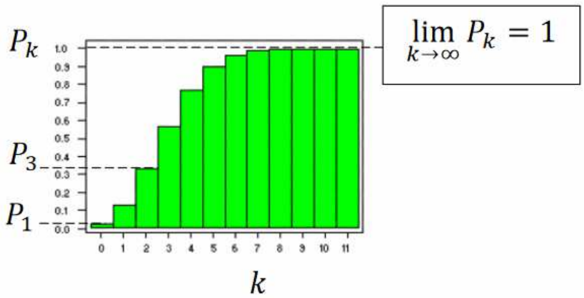
$$cosx = \sum_{k=0}^{\infty} (-1)^k \frac{x^{2k}}{(2k)!} = 1 - \frac{x^2}{2} + \frac{x^4}{24} \pm \cdots, \quad x \in \mathbb{R}$$

Statistics: Cumulative Distributions of Probability, Discrete Variable

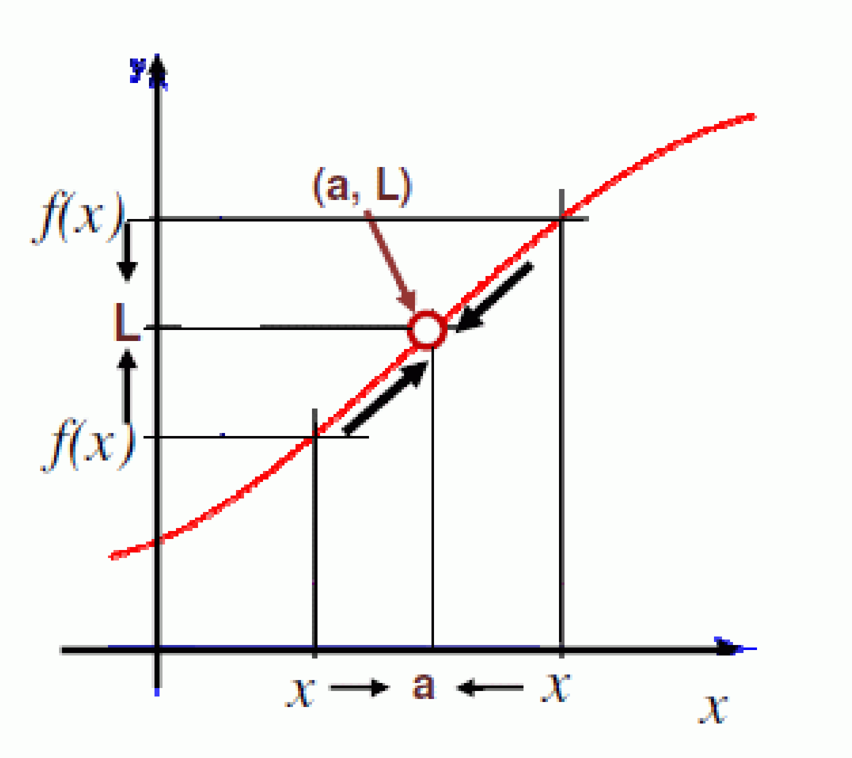
$$P_k = \sum_{l=0}^{k} p_l \,, k = 0,1,2, \ldots$$

$$\lim_{k \to \infty} P_k = 1$$





2. Limits of functions



Informal Definition:
If the values of $f(x)$ can be made as close to L as we like by taking values of x sufficiently close to a [but not equal to a] **then** we write

$$\lim_{x\to a} f(x) = L$$

or $f(x) \to L$ as $x \to a$

Observe:

- "$x \to a$" means x can approach a **from either side**
- On a sketch, the graph of $f(x)$ approaches the 2-D plane location [destination] called (a, L), but the graph itself may have no point $(a, f(a))$ occupying that location!

L may not be $f(a)$

Notation to describe this behaviour of *f* when the input *x* approaches the *x*-value *a*:

$$\lim_{x\to a} f(x) = L$$

Example:

$$\lim_{x \to 0} \left[\frac{x}{\sqrt{x+1} - 1} \right]$$

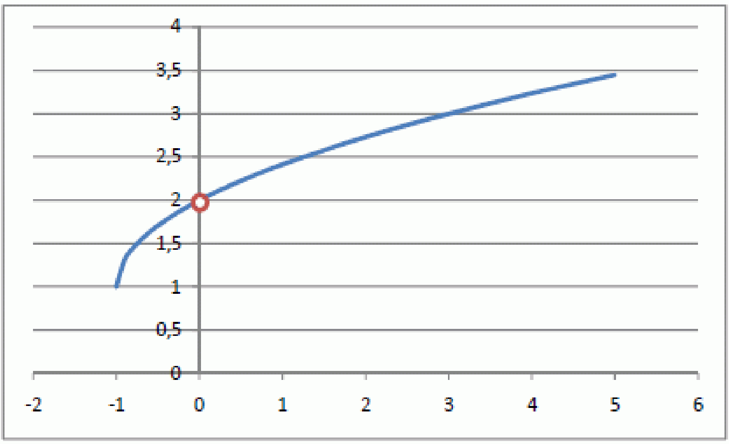
Domain of $f(x)$

$$\sqrt{x+1} - 1 \neq 0, x \neq 0$$

$$x + 1 \geq 0, x \geq -1$$

$$\{x \in \mathbb{R} | x \geq -1, x \neq 0\}$$

Graph of f:



Conjecture:

$$\lim_{x \to 0} \left[\frac{x}{\sqrt{x+1} - 1} \right] = 2$$

(The methods how to prove this will be introduced later.)

General definition:

Let $f(x)$ be a function and a a real number (that may be or may be not in the **domain of** $\boldsymbol{f}$). We say that the limit as x approaches a of $f(x)$ is L, written

$$\lim_{x\to a} f(x) = L$$

if $f(x)$ can be made arbitrarily close to L by choosing x sufficiently close to (but not equal to) a.

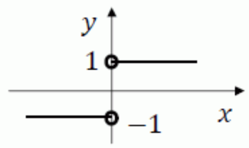
If no such number exists, then we say that

$\lim_{x\to a} f(x)$ does not exist .

Warning: Not all limits exist!

Example:

$$f(x) = \frac{|x|}{x} = \begin{cases} 1 \; if \; x > 0 \\ -1 \; if \; x < 0 \end{cases}$$



- $x \to 0$ from the left, $f(x) \to -1$
- $x \to 0$ from the right, $f(x) \to 1$

So $\lim_{x\to 0} f(x)$ has no meaning!

Two-Sided and One-Sided Limits

Notation

"x approaches a from the left"

$x \to a^-$ [minus in a superscript position] or $x \uparrow a$ [comes up to a] or $x \nearrow a$

$$\lim_{x\to a^-} f(x) = L$$

"x approaches a from the right"

$x \to a^+$ [plus in a superscript position] or $x \downarrow a$ [comes down to a] or $x \searrow a$

$$\lim_{x\to a^+} f(x) = L$$

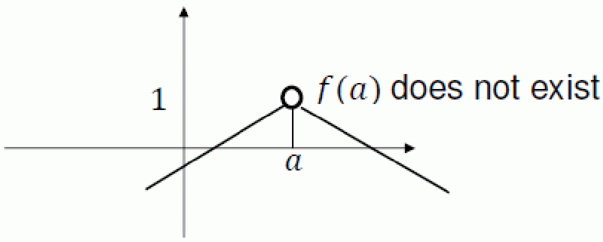
Relationship between Two-Sided and One-Side Limits: Theorem

$$\lim_{\underbrace{x\to a}_{\text{two-sided}}} f(x) = L$$

$\Leftrightarrow$ [if and only if]:

- $\lim_{x\to a^-} f(x)$ exists
- $\lim_{x\to a^+} f(x)$ exists
- and both equal L

Example:



$$\lim_{x\to a^-} f(x) = 1 = \lim_{x\to a^+} f(x)$$

Limits of some basic functions for $x \to a$:

The constant function

$$\lim_{x\to a}(k) = k$$

The identity function: $f(x)$

$$\lim_{x\to a}(x) = a$$

The reciprocal ("flip over") function: $f(x) = \frac{1}{x}$

$$\lim_{x\to 0^-}\left(\frac{1}{x}\right) = -\infty$$

$$\lim_{x\to 0^+}\left(\frac{1}{x}\right) = \infty$$

Limits of Sums, Differences, Products, Quotients and Roots

The "Rules" of Algebra for Limits

Let a be any real number and

$$\lim_{x \to a} f(x) = L_1$$

$$\lim_{x \to a} g(x) = L_2$$

then

$$\lim_{x \to a}[f(x) + g(x)] = \lim_{x \to a} f(x) + \lim_{x \to a} g(x) = L_1 + L_2$$

$$\lim_{x \to a}[f(x) - g(x)] = \lim_{x \to a} f(x) - \lim_{x \to a} g(x) = L_1 - L_2$$

$$\lim_{x \to a}[f(x) \cdot g(x)] = \lim_{x \to a} f(x) \cdot \lim_{x \to a} g(x) = L_1 \cdot L_2$$

$$\lim_{x \to a}|f(x)| = \left|\lim_{x \to a} f(x)\right| = |L_1|$$

$$\lim_{x \to a}[kf(x)] = k \cdot \lim_{x \to a} f(x) = k \cdot L_1$$

$$\lim_{x \to a}[f(x)^n] = \left[\lim_{x \to a} f(x)\right]^n = \mathrm{L}_1^n$$

$$\lim_{x \to a}\left[\frac{f(x)}{g(x)}\right] = \frac{\lim_{x \to a} f(x)}{\lim_{x \to a} g(x)} = \frac{L_1}{L_2}$$

Provided $L_2 \neq 0$

$$\lim_{x \to a} \sqrt[n]{f(x)} = \sqrt[n]{\lim_{x \to a} f(x)} = \sqrt[n]{L_1}$$

Provided when n= even then $L_1 \geq 0$

Limits of polynomial functions

Polynomial Expressions

A monomial (one-term polynomial) has the form

$$a_n \cdot x^n$$

n=0,1,2, 3,…
not negative
called the **degree** of the monomial

A real number constant called a "coefficient"
Subscript n is a label

x- a variable

Two monomials with the same degree and the same variable are called "*like terms*".
$a_n x^n$ and $b_n x^n$ are "like terms".

A polynomial in one variable has the standard form: [higher powers → lower powers]

$$a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$$

$a_n \neq 0$ leading coefficient

Limits of polynomials

Example $\lim_{x\to 5} (x^2 - 4x + 3)$:

By the "Rules of Algebra" for Limits we can break down **polynomials** into simpler parts

Example:

$$\lim_{x\to a}[f(x) \pm g(x)] = \lim_{x\to a} f(x) \pm \lim_{x\to a} g(x)$$

$$\lim_{x\to a}[f(x)^n] = \left[\lim_{x\to a} f(x)\right]^n$$

$$\lim_{x\to a}(k) = k$$

$$\lim_{x\to a}(x) = a$$

$$\lim_{x\to 5}[x^2 - 4x + 3] = \lim_{x\to 5}[x^2] - \lim_{x\to 5}[4x] + \lim_{x\to 5}[3] = \left(\lim_{x\to a}[x]\right)^2 - 4\lim_{x\to a}[x] + 3 =$$

$$\lim_{x\to a}[kf(x)] = k \cdot \lim_{x\to a} f(x)$$

$$= 5^2 - 4 \cdot 5 + 3 = 8$$

For any polynomial function

$$\lim_{x \to a} p(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0 = p(a)$$

This result is the same as the result of "substituting *a* for *x*" in the polynomial.

Thus, the calculation of limits of polynomial functions is "easy": Just insert *a* for *x*.

Limits of Rational Functions $\frac{p(x)}{q(x)}$ and the appearance of $\frac{0}{0}$

There are 3 cases to consider

Case 1: $q(a) \neq 0$ Limit $= \frac{p(a)}{q(a)}$

Example:

$$\lim_{x \to 2} \left[\frac{5x^3 + 4}{x - 3} \right] \begin{matrix} \leftarrow p(x) \\ \leftarrow q(x) \end{matrix} \quad a = 2$$

$$= \frac{\lim\limits_{x \to 2} [5x^3 + 4]}{\lim\limits_{x \to 2} [x - 3]} = \frac{\overbrace{5 \cdot 2^3 + 4}^{p(a)}}{\underbrace{2 - 3}_{q(a) \neq 0}} = -44$$

$\rightarrow$ this case is also "easy", same as for polynomial functions.

Case 2: $p(a) \neq 0 \; and \; q(a) = 0$ Limit does not exist (division by 0!)

$$f(x) = \frac{\overbrace{1}^{p(x)}}{\underbrace{x-a}_{q(x)}}$$

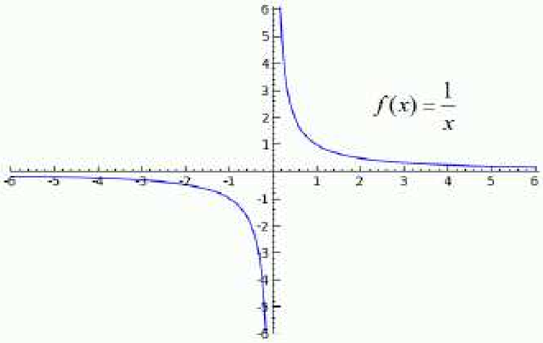
$$\lim_{x \to a^-} \left[\frac{1}{x-a}\right] = -\infty; \; \lim_{x \to a^+} \left[\frac{1}{x-a}\right] = +\infty$$

Classic Examples:

$$f(x) = \frac{1}{x}$$

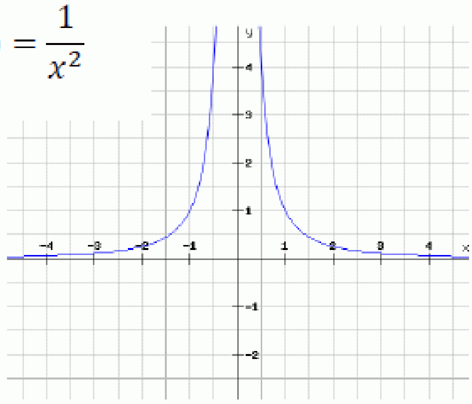
$$\lim_{x \to 0^-} \left[\frac{1}{x}\right] = -\infty$$

$$\lim_{x \to 0^+} \left[\frac{1}{x}\right] = +\infty$$



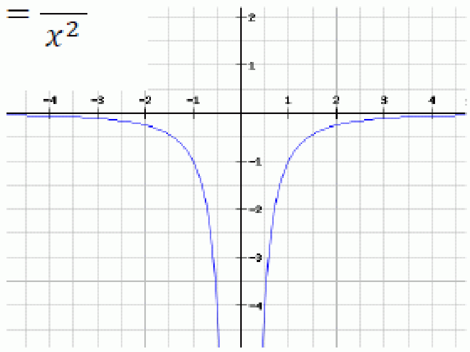
$$f(x) = \frac{1}{x^2}$$

$$\lim_{x \to a} \left[\frac{1}{(x-a)^2}\right] = \infty$$



$$f(x) = \frac{-1}{x^2}$$

$$\lim_{x \to a} \left[\frac{-1}{(x-a)^2}\right] = -\infty$$



Case 3: $p(a) = 0 \text{ and } q(a) = 0$ Limit $\frac{p(a)}{q(a)} = \frac{0}{0}$ an indeterminate form: We cannot determine whether the limit exists or not, without more work!

Example:

$$\lim_{x \to 2} \left[\frac{x^2 - 4}{x - 2} \begin{matrix} \leftarrow p(x) \\ \leftarrow q(x) \end{matrix} \right] \text{both } p(2) = 0 = q(2)$$

Factor + cancel

$$\lim_{x \to 2} \left[\frac{(x-2)(x+2)}{x-2} \right] = \lim_{x \to 2} [(x+2)] = 4$$

This is only one particularly technique! Does not work always!

A more general method to solve this case will be introduced later.

Until now, we have considered only the case that *x* approaches some real number *a*. But often one is interested what happens with *f*(*x*) when *x* gets smaller and smaller, or larger and larger.
We write $x \to \pm\infty$ for this case, and speak of "end behavior" of *f*.

The Algebra of Limits as $x \to \pm\infty$: End Behavior

Basic Limits:

The constant function

$$\lim_{x \to -\infty} (k) = k$$

and

$$\lim_{x \to \infty} (k) = k$$

The identity function: $f(x)$

$$\lim_{x \to -\infty} (x) = -\infty$$

$$\lim_{x \to \infty} (x) = \infty$$

The reciprocal ("flip over") function: $f(x) = \frac{1}{x}$

$$\lim_{x \to -\infty} \left(\frac{1}{x}\right) = 0$$

$$\lim_{x \to \infty} \left(\frac{1}{x}\right) = 0$$

Limits of Sums, Differences, Products, Quotients and Roots

The "Rules" of Algebra for Limits applied to $x \to -\infty$ or $x \to \infty$

We only state for $x \to \infty$ case

As before, suppose:

$$\lim_{x \to \infty} f(x) = L_1$$

$$\lim_{x \to \infty} g(x) = L_2$$

then

$$\lim_{x \to \infty} [f(x) + g(x)] = \lim_{x \to \infty} f(x) + \lim_{x \to \infty} g(x) = L_1 + L_2$$

$$\lim_{x \to \infty} [f(x) - g(x)] = \lim_{x \to \infty} f(x) - \lim_{x \to \infty} g(x) = L_1 - L_2$$

$$\lim_{x \to \infty} [f(x) \cdot g(x)] = \lim_{x \to \infty} f(x) \cdot \lim_{x \to \infty} g(x) = L_1 \cdot L_2$$

$$\lim_{x \to \infty} |f(x)| = \left|\lim_{x \to \infty} f(x)\right| = |L_1|$$

$$\lim_{x \to \infty} [kf(x)] = k \cdot \lim_{x \to \infty} f(x) = k \cdot L_1$$

$$\lim_{x \to \infty} [f(x)^n] = \left[\lim_{x \to \infty} f(x)\right]^n = \mathrm{L}_1^n$$

$$\lim_{x \to \infty} \left[\left(\frac{1}{x}\right)^n\right] = \left[\lim_{x \to \infty} \frac{1}{x}\right]^n = 0$$

$$\lim_{x \to \infty} \left[\frac{f(x)}{g(x)}\right] = \frac{\lim_{x \to \infty} f(x)}{\lim_{x \to \infty} g(x)} = \frac{L_1}{L_2}$$

Provided $L_2 \neq 0$

$$\lim_{x \to \infty} \sqrt[n]{f(x)} = \sqrt[n]{\lim_{x \to \infty} f(x)} = \sqrt[n]{L_1}$$

Provided when n= even then $L_1 \geq 0$

Limits of Polynomial Functions: Two End Behaviors

A polynomial function

$$f(x) = c_n x^n + c_{n-1}x^{n-1} + \cdots + c_1 x + c_0$$

Where $c_n \neq 0$

The "two end behaviors" are that as $x \to \infty$ (the rightward end) or $x \to -\infty$ (the leftward end)

Then

$$\left.\begin{matrix} f(x) \to \infty \\ f(x) \to -\infty \end{matrix}\right\} \text{ The two possibilities}$$

Observe:

$$f(x) = c_n x^n + c_{n-1}x^{n-1} + c_1 x + c_0 = x^n \left[c_n + \underbrace{\frac{c_{n-1}}{x} + \cdots + \frac{c_1}{x^{n-1}} + \frac{c_0}{x^n}}_{go\ to\ 0} \right]$$

So, the "end behavior" of $f(x)$ matches the "end behavior" of $c_n x^n$

Theorem:

$$\lim_{x\to\pm\infty} [c_n x^n + c_{n-1}x^{n-1} + c_1 x + c_0] = \lim_{x\to\pm\infty} [c_n x^n]$$

Example:

$$\lim_{x\to-\infty} [-4x^8 + 17x^5 + 3x^4 + 2x - 50] = \lim_{x\to-\infty} [-4x^8] = -\infty$$

Limits of Rational Functions: Three Types of End Behavior

$$f(x) = \frac{p(x)}{q(x)} \begin{matrix} \leftarrow top \\ \leftarrow botton \end{matrix}$$

Remember: The Degree of a polynomial is the exponent of the highest power of x in the polynomial

Type 1. Deg(top)=Deg(bottom)

$$\lim_{x\to\infty} f(x) = \frac{leading\ coefficient\ of\ top}{leading\ coefficient\ of\ bottom}$$

Example:

$$f(x) = \frac{-x}{7x+4}$$

$$\lim_{x\to\infty} f(x) = \lim_{x\to\infty}\left[\frac{-x}{7x+4}\right] = \lim_{x\to\infty}\left[\frac{-1}{7+\underbrace{\frac{4}{x}}_{\searrow 0}}\right] = -\frac{1}{7} = \frac{l.c.of\ top}{l.c.of\ bottom}$$

(**Deg** = degree)

Type 2. Deg(top)<Deg(bottom)

$$\lim_{x\to\infty} f(x) = 0$$

Example:

$$f(x) = \frac{5x+2}{2x^3-1}$$

$$\lim_{x\to\infty} f(x) = \lim_{x\to\infty}\left[\frac{5x+2}{2x^3-1}\right] = \lim_{x\to\infty}\left[\frac{\frac{5}{x^2}+\frac{2}{x^3}}{2-\frac{1}{x^3}}\right] = 0$$

Always zero

$y = 0$ the (x-axis) is a horizontal asymptote

Type 3. Deg(top)>Deg(bottom)

If $leading\ coefficient\ of\ top > 0$

$$\left.\begin{array}{lll} \infty & if & x \to \infty \\ -\infty & if & x \to -\infty \end{array}\right\}$$ Always one of these

If $leading\ coefficient\ of\ top < 0$

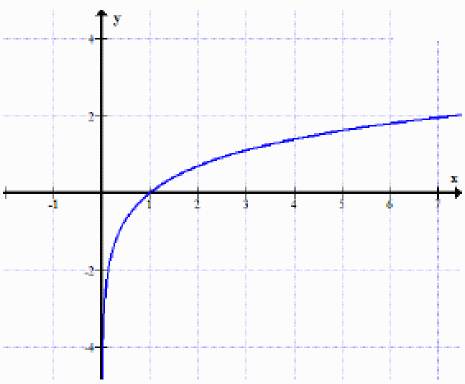
$$\left.\begin{array}{lll} \infty & if & x \to -\infty \\ -\infty & if & x \to \infty \end{array}\right\}$$ Always one of these

Example:

$$f(x) = \frac{x^2 + 4x + 5}{x - 1}$$

$$\lim_{x\to\infty} f(x) = \lim_{x\to\infty}\left[\frac{x^2 + 4x + 5}{x - 1}\right] = \lim_{x\to\infty}\left[\frac{1 + \frac{4}{x} + \frac{5}{x^2}}{\frac{1}{x} - \frac{1}{x^2}}\right] = \infty$$

Limits of ln(x)

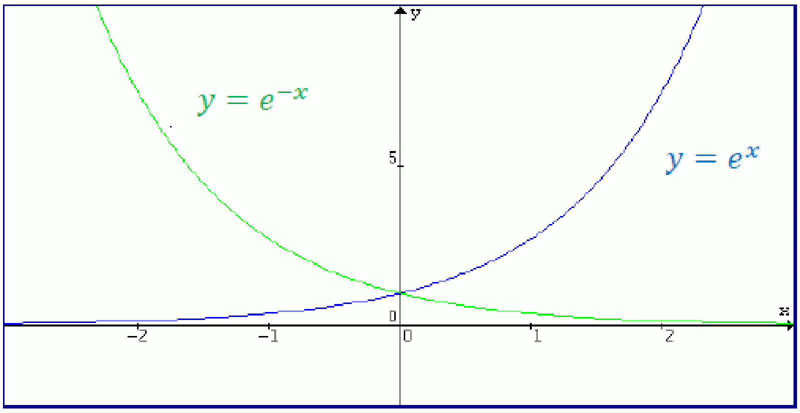


$$\lim_{x\to\infty}[lnx] = \infty$$

note, that $\lim_{x\to-\infty} lnx$ makes no sense

$$\lim_{x\to0^+}[lnx] = -\infty$$

Limits of e^x



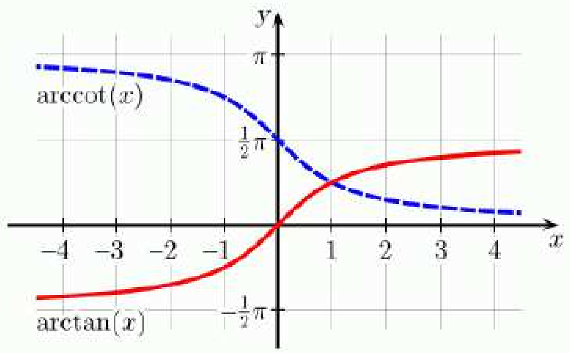
$$\lim_{x \to \infty} [e^x] = \infty$$

$$\lim_{x \to -\infty} [e^x] = 0$$

$$\lim_{x \to \infty} [e^{-x}] = 0$$

$$\lim_{x \to -\infty} [e^{-x}] = \infty$$

Limits of $arctanx$ and $arccotx$



$$\lim_{x \to \infty} arctanx = \frac{\pi}{2}$$

$$\lim_{x \to -\infty} arctanx = -\frac{\pi}{2}$$

$$\lim_{x \to \infty} arccotx = 0$$

$$\lim_{x \to -\infty} arccotx = \pi$$

Sources:
Irina Kuzyakova: Computer Science and Mathematics (study course MES), summer semester 2014, part "Basics of Calculus"
Richard Delaware (Univ. of Missouri): Lectures (youtube.com)
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Wikipedia