Exercises 13 — Solutions

Task 1

$f(x,y) = 18x^2 - 36xy - 6y^3 - 36x + 20y$

(a) $f_x(x,y) = 36x - 36y - 36$

$f_y(x,y) = -36x - 18y^2 + 20$

$f_{xx}(x,y) = 36$

$f_{xy}(x,y) = -36$

$f_{yy}(x,y) = -36y$

(b) $f_x(x,y) = 0 \iff 36x - 36y = 36 \iff y = x - 1$

$f_y(x,y) = 0 \iff 18x + 9y^2 = 10 \iff 18x + 9(x-1)^2 = 10 \iff 9x^2 = 1$

$\iff x = \pm \frac{1}{3}$; critical points: $\left(\frac{1}{3}, -\frac{2}{3}\right); \left(-\frac{1}{3}, -\frac{4}{3}\right)$

$\begin{vmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{vmatrix} = \begin{vmatrix} 36 & -36 \\ -36 & -36y \end{vmatrix} = 36^2 \cdot \begin{vmatrix} 1 & -1 \\ -1 & -y \end{vmatrix} = -36^2 \cdot (y+1) \overset{!}{>} 0 \iff y < -1$

extremal point in $\left(-\frac{1}{3}, -\frac{4}{3}\right)$

$y = -\frac{2}{3}$ does not fulfill this, no extremum in $\left(\frac{1}{3}, -\frac{2}{3}\right)$

$f_{xx}(x,y) = 36 > 0 \implies$ it must be a local minimum

Task 2

(a) $\int (5x^4 - 3 + \frac{2}{\sqrt{x}}) dx = x^5 - 3x + 4\sqrt{x} + C$

(Formula collection)

(b) $\int \frac{4x^3 - \frac{1}{x}}{x^4 - \ln x} dx = \int \frac{u'(x)}{u(x)} dx \ [\text{with } u(x) = x^4 - \ln x] = \ln |u(x)| + C = \ln |x^4 - \ln x| + C$

(c) $\int \frac{7}{\sqrt{4-t^2}} dt = 7 \cdot \arcsin \frac{t}{2} + C$ (Table!)

(d) $\int 2^{x-1} dx = \frac{1}{\ln 2} \cdot 2^{x-1} + C$

(e) $\int \frac{2}{\cos^2(2x)} dx = \tan(2x) + C$

Task 3

(a) $\int_0^1 (e^x + \frac{3}{2}x^2) dx = \left[e^x + \frac{1}{2}x^3 \right]_0^1 = e + \frac{1}{2} - 1 = e - \frac{1}{2}$

(b) $\int_{-\pi/4}^{\pi/4} \sin x \, dx = \left[-\cos x \right]_{-\frac{\pi}{4}}^{\frac{\pi}{4}} = -\cos \frac{\pi}{4} + \cos\left(-\frac{\pi}{4}\right) = -\cos \frac{\pi}{4} + \cos \frac{\pi}{4} = 0$

(c) $\int_{41}^{42} x \, dx = \left[\frac{1}{2}x^2 \right]_{41}^{42} = \frac{1}{2} \cdot 42^2 - \frac{1}{2} \cdot 41^2 = \frac{1}{2}(1764 - 1681) = \frac{1}{2} \cdot 83 = 41.5$

(d) $\int_{-1}^1 \frac{1}{9-x^2} dx = \left[\frac{1}{6} \ln \left| \frac{3+x}{3-x} \right| \right]_{-1}^1 = \frac{1}{6}\left(\ln \left|\frac{4}{2}\right| - \ln \left|\frac{2}{4}\right| \right) = \frac{1}{6}\left(\ln 2 - \ln \frac{1}{2} \right)$

↑ Table!

$= -\ln 2$

$= \frac{1}{6} \cdot 2 \ln 2 = \frac{1}{3} \ln 2 \ (\approx 0.231)$

Task 4

area $= \int_1^3 (f-g)(x) \, dx = \int_1^3 (4x - x^2 - 3 - (x-3)) dx = \int_1^3 (-x^2 + 3x) dx = \left[-\frac{1}{3}x^3 + \frac{3}{2}x^2 \right]_1^3$

$= -9 + \frac{27}{2} + \frac{1}{3} - \frac{3}{2} = 3 + \frac{1}{3} = \frac{10}{3}$