This file is available for download on:	*StudIP (restricted access)*	https://studip-ecampus.uni-goettingen.de/
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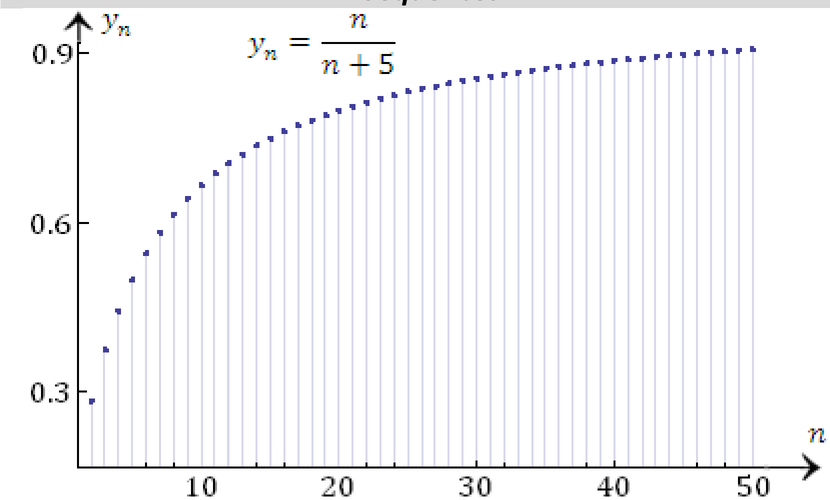
***Sneak Peek* into the topics:**

Sequences	***Series***
	$$\sum_{n=1}^{\infty}\frac{2n-1}{n(1+n)(n+2)}\sim\sum_{n=1}^{\infty}\frac{2n}{n^3}=\sum_{n=1}^{\infty}\frac{2}{n^2}$$

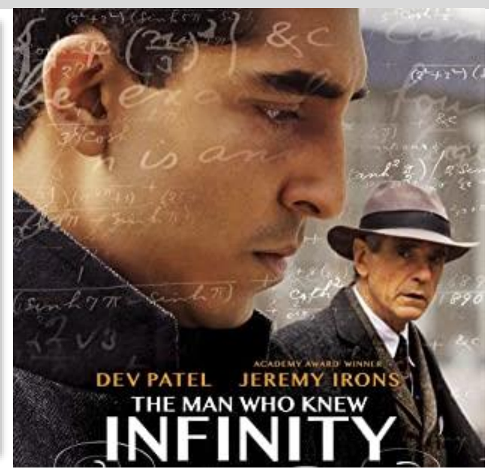
Infinity

XI. I have got theorems on divergent series, theorems to calculate the convergent values corresponding to the divergent series, viz.

$$1-2+3-4+\cdots=\frac{1}{4},$$

$$1-1!+2!-3!+\cdots=.596\ldots,$$

$$1+2+3+4+\cdots=-\frac{1}{12}$$



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Sequences: *task #1*

Task description

Prove the following implication (aka "**IF-THEN**" statement):

$$\begin{cases} x_1 = \frac{1}{2} \\ x_{n>1} = x_{n-1} + \frac{1}{2^n} \end{cases} \Longrightarrow x_n = 1 - \frac{1}{2^n}$$

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What you need to know:

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Solution:

All we need to do is to convert the recursive formula (i.e. the **LHS** of our implication) into the closed-form and to check if it coincides with the RHS:

$$x_n = x_{n-1} + \frac{1}{2^n} = x_{(n-1)-1} + \frac{1}{2^{n-1}} + \frac{1}{2^n} = x_{n-2} + \frac{1}{2^{n-1}} + \frac{1}{2^n} =$$

$$x_{(n-2)-1} + \frac{1}{2^{n-2}} + \frac{1}{2^{n-1}} + \frac{1}{2^n} = x_{n-3} + \frac{1}{2^{n-2}} + \frac{1}{2^{n-1}} + \frac{1}{2^n} =$$

$$x_{(n-3)-1} + \frac{1}{2^{n-3}} + \frac{1}{2^{n-2}} + \frac{1}{2^{n-1}} + \frac{1}{2^n} = x_{n-4} + \frac{1}{2^{n-3}} + \frac{1}{2^{n-2}} + \frac{1}{2^{n-1}} + \frac{1}{2^n} =$$

$$\dots$$

$$x_{n-(n-1)} + \frac{1}{2^{n-(n-1)+1}} + \dots + \frac{1}{2^{n-3}} + \frac{1}{2^{n-2}} + \frac{1}{2^{n-1}} + \frac{1}{2^n} =$$

$$x_1 + \frac{1}{2^2} + \dots + \frac{1}{2^{n-3}} + \frac{1}{2^{n-2}} + \frac{1}{2^{n-1}} + \frac{1}{2^n} =$$

$$\frac{1}{2} + \frac{1}{2^2} + \dots + \frac{1}{2^{n-3}} + \frac{1}{2^{n-2}} + \frac{1}{2^{n-1}} + \frac{1}{2^n} = \frac{1}{2^1} + \frac{1}{2^2} + \dots + \frac{1}{2^{n-3}} + \frac{1}{2^{n-2}} + \frac{1}{2^{n-1}} + \frac{1}{2^n} =$$

$$first_{term} \cdot \frac{1 - common_{ratio}{}^n}{1 - common_{ratio}} = \frac{1}{2^1} \cdot \frac{1 - \left(\frac{1}{2}\right)^n}{1 - \frac{1}{2}} = \frac{1}{2} \cdot \frac{1 - \frac{1}{2^n}}{\frac{1}{2}} = 1 - \frac{1}{2^n}$$

Sum of the first n terms of a geometric series

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Sequences - limits: *task #2*

Task description

Check if:

$$\left(-\frac{2}{3}\right)^{n} \longrightarrow 0 \text{ as } n \longrightarrow \infty$$

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What you need to know:

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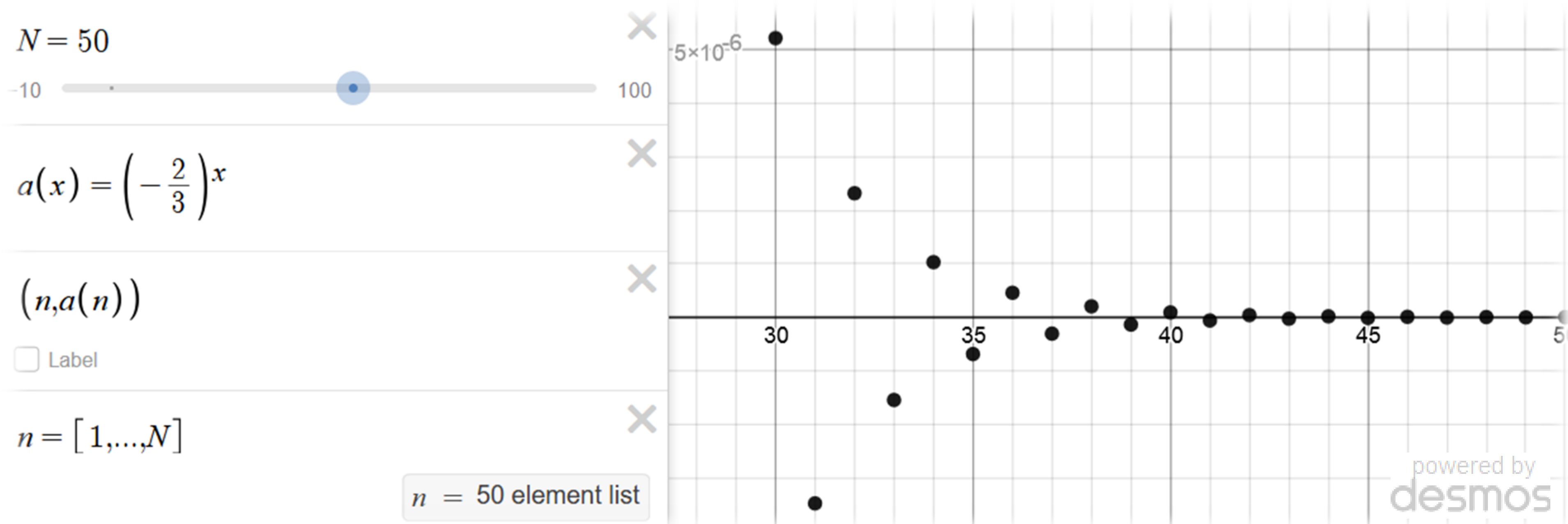
Solution:

Plotting target functions and observing their behaviour visually **can never go wrong** (well, unless you plotted them erroneously ☺):



As the plot immediately shows, our function (**reminder**: a sequence is a function over natural numbers – in other words, a discrete function) mercilessly ☺ **tends to 0 as n increases** [check out the scale on the Y-axis – we are literally in the **micro-neighbourhood of 0**, starting already from the 30^{th} term of our sequence].

So, we do get a very strong feeling that the required check has to succeed!

Of course, just by graphing functions, you ain't proving anything in terms of rigid mathematics – so, we really have to proceed with the proof ☺:

As we already know, all we wanna do is to solve the following inequality for $\boldsymbol{n}$:

$$\left|\left(-\frac{2}{3}\right)^{n} - 0\right| < \varepsilon$$

Let's simplify it a lil bit:

$$\left|\left(-\frac{2}{3}\right)^{n} - 0\right| < \varepsilon \Leftrightarrow \left|\left(-\frac{2}{3}\right)^{n}\right| < \varepsilon \Leftrightarrow \left(\frac{2}{3}\right)^{n} < \varepsilon$$

So, we've reduced out task to:

$$\left(\frac{2}{3}\right)^{n} < \varepsilon$$

As soon as $\boldsymbol{n}$ is an exponent here, solving for it requires the usage of logarithms – let's take, e.g., a natural one, $\boldsymbol{ln}$:

$$\left(\frac{2}{3}\right)^{n} < \varepsilon \Leftrightarrow ln\left(\frac{2}{3}\right)^{n} < ln(\varepsilon)$$

Using some basic properties of logarithmic functions, we end up with:

$$ln\left(\frac{2}{3}\right)^{n} < ln(\varepsilon) \Leftrightarrow n \cdot ln\left(\frac{2}{3}\right) < ln(\varepsilon)$$

Observe that $\boldsymbol{ln}\left(\frac{2}{3}\right) < \boldsymbol{0}$ – this yields:

For those who might wanna take a glimpse into the **basics of solving inequalities**: https://www.mathsisfun.com/algebra/inequality-solving.html

$$n > \frac{ln(\varepsilon)}{ln\left(\frac{2}{3}\right)} = n(\varepsilon)$$

Example: let's say, for $\boldsymbol{\varepsilon = 10^{-6}}$ we get $\boldsymbol{n} > \frac{ln(\varepsilon)}{ln\left(\frac{2}{3}\right)} = \frac{ln(10^{-6})}{ln\left(\frac{2}{3}\right)} \approx \boldsymbol{34.1}$ – so, starting from its $\boldsymbol{35^{th}}$ term, our sequence gets at least 10^{-6}-close to 0.

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Infinite Series: *task #3*

Task description

Compute the following series:

$$\sum_{k=2}^{\infty} \frac{2}{k^2 - 1}$$

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What you need to know:

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Solution:

First of all, notice that:

$$\frac{2}{k^2-1}=\frac{2}{k^2-1^2}=\frac{2}{(k-1)\cdot(k+1)}=\frac{(k+1)-(k-1)}{(k-1)\cdot(k+1)}=\frac{(k+1)}{(k-1)\cdot(k+1)}-\frac{(k-1)}{(k-1)\cdot(k+1)}=\frac{1}{k-1}-\frac{1}{k+1}$$

This turns our target series into an easy-to-compute **telescopic series**:

$$\sum_{k=2}^{\infty}\frac{2}{k^2-1}=\sum_{k=2}^{\infty}\left(\frac{1}{k-1}-\frac{1}{k+1}\right)=\lim_{n\to\infty}\sum_{k=2}^{n}\left(\frac{1}{k-1}-\frac{1}{k+1}\right)=\lim_{n\to\infty}\left(\sum_{k=2}^{n}\left(\frac{1}{k-1}\right)-\sum_{k=2}^{n}\left(\frac{1}{k+1}\right)\right)=$$

$$\lim_{n\to\infty}\left(\frac{1}{2-1}+\frac{1}{3-1}+\sum_{k=4}^{n}\left(\frac{1}{k-1}\right)-\sum_{k=2}^{n-2}\left(\frac{1}{k+1}\right)-\frac{1}{(n-1)+1}-\frac{1}{n+1}\right)=$$

$$\lim_{n\to\infty}\left(\frac{1}{1}+\frac{1}{2}+\sum_{k=4-2}^{n-2}\left(\frac{1}{(k-1)+2}\right)-\sum_{k=2}^{n-2}\left(\frac{1}{k+1}\right)-\frac{1}{n}-\frac{1}{n+1}\right)=$$

$$\lim_{n\to\infty}\left(\frac{1}{1}+\frac{1}{2}+\sum_{k=2}^{n-2}\left(\frac{1}{k+1}\right)-\sum_{k=2}^{n-2}\left(\frac{1}{k+1}\right)-\frac{1}{n}-\frac{1}{n+1}\right)=$$

$$\lim_{n\to\infty}\left(\frac{1}{1}+\frac{1}{2}-\frac{1}{n}-\frac{1}{n+1}\right)=\frac{1}{1}+\frac{1}{2}-\lim_{n\to\infty}\left(\frac{1}{n}+\frac{1}{n+1}\right)=\frac{1}{1}+\frac{1}{2}-0=\frac{3}{2}$$

Homework: check if $\lim_{n\to\infty}\left(\frac{1}{n}+\frac{1}{n+1}\right)=0$

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Infinite Series: *task #4*

Task description

Check whether the following series converge or diverge:

A	$$\sum_{k=1}^{\infty} \frac{k}{k+1}$$	**C**	$$\sum_{k=1}^{\infty} \frac{3^k}{4^k+4}$$
B	$$\sum_{k=1}^{\infty} \frac{k}{2^k}$$	**D**	$$\sum_{k=1}^{\infty} \frac{k+2}{k^2+3}$$

Hint: don't forget to take a look at the Cheat Sheet family on **Convergence**

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What you need to know:

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Solution:

A

Let's pick the Divergence Test:

So, the thing to check is if the following holds:

$$\lim_{k\to\infty}\frac{k}{k+1}\neq 0$$

Check it:

That's your homework, folks ☺

Expected test outcome is: affirmative (i.e. series **diverge**)

B

Let's pick the Geometric Series Test and the Direct Comparison Test (their **Conditions of Convergence**), because:

We could easily notice that:

$$\frac{3^k}{4^k+4}<\frac{3^k}{4^k}=\left(\frac{3}{4}\right)^k$$

Another (and last) thing to notice is that:

The rest here is also your homework ☺

Expected test outcomes are: affirmative (i.e. series **converge**)

C

Let's pick the **Ratio Test** (its **Condition on Convergence**):

So, the thing to check is whether the following holds:

$$\lim_{k\to\infty}\left|\frac{\frac{k+1}{2^{k+1}}}{\frac{k}{2^k}}\right| < 1$$

Check it:

That's your homework ☺

Expected test outcome is: affirmative (i.e. series **converge**)

D

Let's pick the **p-Series Test** and the **Limit Comparison Test** (its **Condition on Divergence**) and:

Get the things done ☺ (homework)

Expected test outcome is: affirmative (i.e. series **diverge**)

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Homework Assignments:

are **exam-relevant**, and if completed & submitted/shown prior to the next week class sessions (either in written or oral form), could bring **bonus points** for the exam

Task #1: **Sequences/Series**

By using either sequences or series, show that:

$$\frac{1}{3} = 0.333333\cdots$$

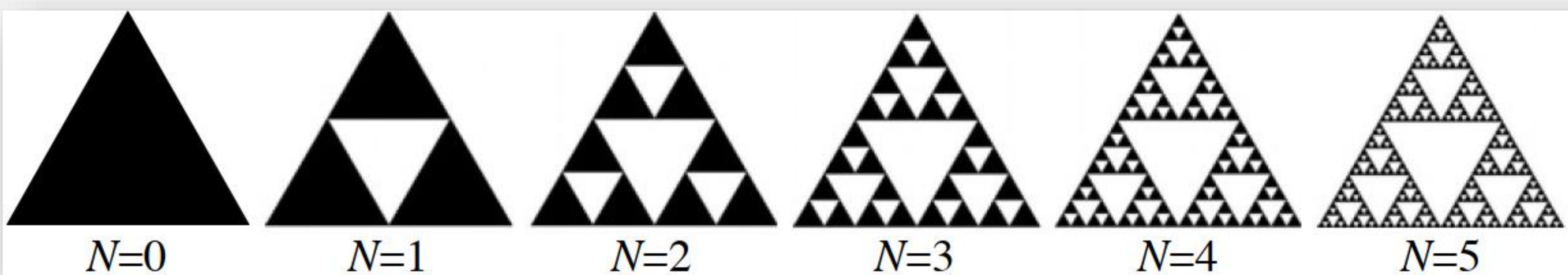
Task #2: **Sequences**

Check what happens with the following sequence of squares:

$$x_n = n^2$$

Task #3: **Series**

Compute the **total shaded area** after n steps of the following procedure:



Remark: all triangles are **equilateral**; the area of the first one is **1**

Leaderboard: bonus points *per capita* ☺, cumulative

Cheat Sheet of the Day ☺: *Convergence Tests*

1 Divergence or *nth* Term Test

Series: $\sum_{n=1}^{\infty} a_n$

Condition(s) of Convergence:
None. This test cannot be used to show convergence.

Condition(s) of Divergence:
$\lim_{n\to\infty} a_n \neq 0$

2 Geometric Series Test

Series: $\sum_{n=0}^{\infty} ar^n$

Condition of Convergence:
$|r| < 1$

Sum: $S = \lim_{n\to\infty} \frac{a(1-r^n)}{1-r} = \frac{a}{1-r}$

Condition of Divergence:
$|r| \geq 1$

3 *p* - Series Test

Series: $\sum_{n=1}^{\infty} \frac{1}{n^p}$

Condition of Convergence:
$p > 1$

Condition of Divergence:
$p \leq 1$

4 Alternating Series Test

Series: $\sum_{n=1}^{\infty} (-1)^{n+1} a_n$

Condition of Convergence:
$0 < a_{n+1} \leq a_n$
$\lim_{n\to\infty} a_n = 0$
or if $\sum_{n=0}^{\infty} |a_n|$ is convergent

Condition of Divergence:
None. This test cannot be used to show divergence.

* Remainder: $|R_n| \leq a_{n+1}$

5 Integral Test

Series: [illegible]
[illegible]en $a_n = f(n)$
and [illegible]) is continuous [illegible] and
decreas[illegible]

C[illegible]tion of Co[illegible]ence:
$\int_1^\infty f$[illegible] converges

Con[illegible] Divergence:
[illegible]$f(x)dx$ dive[illegible]

* Remainder: $0 < R_N \leq \int_N^\infty f(x)dx$

6 Ratio Test

Series: $\sum_{n=1}^{\infty} a_n$

Condition of Convergence:
$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| < 1$

Condition of Divergence:
$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| > 1$

* Test *inconclusive* if
$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = 1$

7 Root Test

Series: $\sum_{n=1}^{\infty} a_n$

Condition of Convergence:
$\lim_{n\to\infty} \sqrt[n]{|a_n|} < 1$

Condition of Divergence:
$\lim_{n\to\infty} \sqrt[n]{|a_n|} > 1$

* Test *inconclusive* if
$\lim_{n\to\infty} \sqrt[n]{|a_n|} = 1$

8 Direct Comparison Test
$(a_n, b_n > 0)$

Series: $\sum_{n=1}^{\infty} a_n$

Condition of Convergence:
$0 < a_n \leq b_n$
and $\sum_{n=0}^{\infty} b_n$ is absolutely convergent

Condition of Divergence:
$0 < b_n \leq a_n$
and $\sum_{n=0}^{\infty} b_n$ diverges

9 Limit Comparison Test
$(\{a_n\}, \{b_n\} > 0)$

Series: $\sum_{n=1}^{\infty} a_n$

Condition of Convergence:
$\lim_{n\to\infty} \frac{a_n}{b_n} = L > 0$
and $\sum_{n=0}^{\infty} b_n$ converges

Condition of Divergence:
$\lim_{n\to\infty} \frac{a_n}{b_n} = L > 0$
and $\sum_{n=0}^{\infty} b_n$ diverges

10 Telescoping Series Test

Series: $\sum_{n=1}^{\infty} (a_{n+1} - a_n)$

Condition of Convergence:
$\lim_{n\to\infty} a_n = L$

Condition of Divergence: None

NOTE:
1) May need to reformat with partial fraction expansion or log rules.
2) Expand first 5 terms. n=1,2,3,4,5.
3) Cancel duplicates.
4) Determine limit L by taking the limit as $n \to \infty$.
5) Sum: $S = a_1 - L$

NOTE: These tests prove convergence and divergence, not the actual limit L or sum **S**.

Sequence: $\lim_{n\to\infty} a_n = L$
$(a_n, a_{n+1}, a_{n+2}, \ldots)$

Series: $\sum_{n=1}^{\infty} a_n = S$
$(a_n + a_{n+1} + a_{n+2} + \cdots)$

Cheat Sheet #2: *Binary Decision Tree for* ***choosing a proper Convergence Test***