This file is available for download on:		
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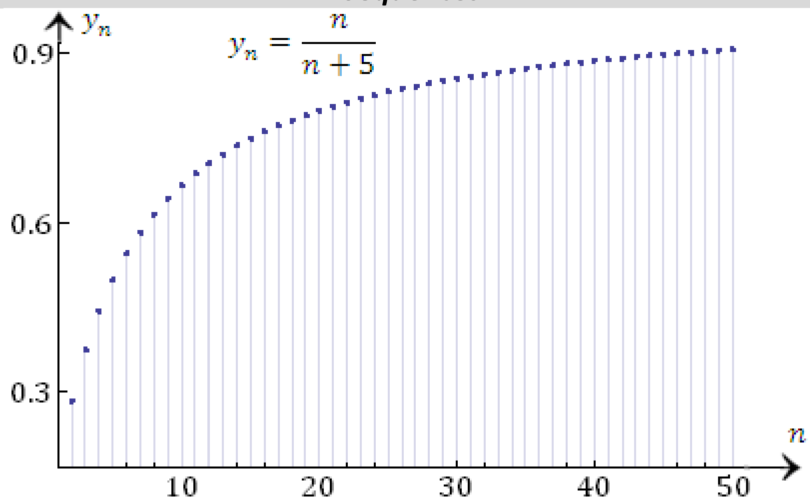
***Sneak Peek* into the topics:**

Sequences	***Series***
	$$\sum_{n=1}^{\infty}\frac{2n-1}{n(1+n)(n+2)}\sim\sum_{n=1}^{\infty}\frac{2n}{n^3}=\sum_{n=1}^{\infty}\frac{2}{n^2}$$

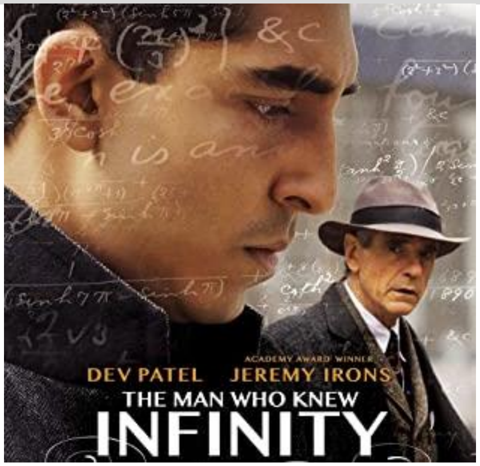
Infinity

XI. I have got theorems on divergent series, theorems to calculate the convergent values corresponding to the divergent series, viz.

$$1-2+3-4+\cdots=\frac{1}{4},$$

$$1-1!+2!-3!+\cdots=.596\ldots,$$

$$1+2+3+4+\cdots=-\frac{1}{12}$$



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Contents

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Sequences: *task #1*

Task description

Write and plot the first **5** terms of the sequence $\{x_n\}_{n\in\mathbb{N}}$ given by:

$$\boldsymbol{x_n = (-1)^{n-1} \cdot \frac{2}{n}}$$

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What you need to know:

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Solution:

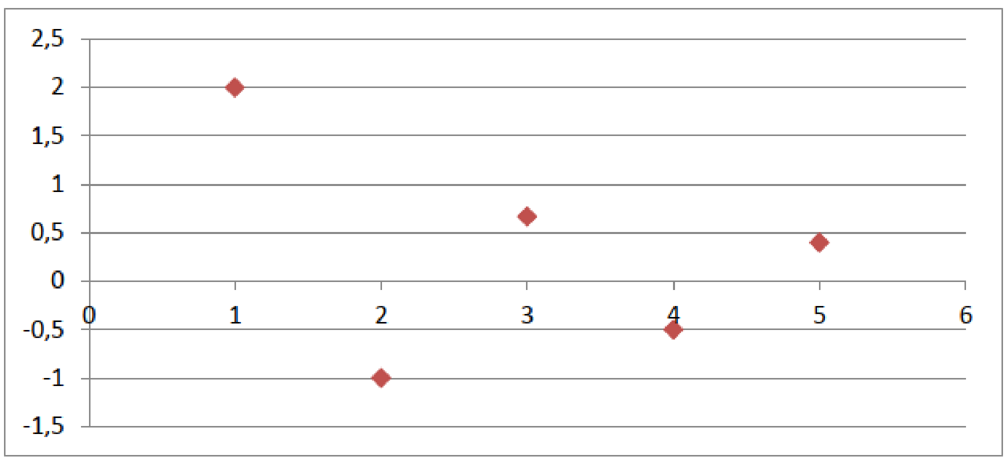
$$x_1 = (-1)^{1-1} \cdot \frac{2}{1} = (-1)^0 \cdot 2 = 1 \cdot 2 = 2$$

$$x_2 = (-1)^{2-1} \cdot \frac{2}{2} = (-1)^1 \cdot 1 = -1 \cdot 1 = -1$$

$$x_3 = (-1)^{3-1} \cdot \frac{2}{3} = (-1)^2 \cdot \frac{2}{3} = 1 \cdot \frac{2}{3} = \frac{2}{3}$$

$$x_4 = (-1)^{4-1} \cdot \frac{2}{4} = (-1)^3 \cdot \frac{1}{2} = -1 \cdot \frac{1}{2} = -\frac{1}{2}$$

$$x_5 = (-1)^{5-1} \cdot \frac{2}{5} = (-1)^4 \cdot \frac{2}{5} = 1 \cdot \frac{2}{5} = \frac{2}{5}$$



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Sequences: *task #2*

Task description

Find a formula for x_n ($n \in \mathbb{N}$) in each of the following cases:

A	$1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \cdots$
B	$2, 2, 2, 0, 0, 0, 0, 0, \cdots$
C	$1, -1, 1, -1, \cdots$
D	$\frac{1}{2}, \frac{3}{4}, \frac{7}{8}, \frac{15}{16}, \cdots$
E	$1, 1, 2, 3, 5, 8, 13, \cdots$

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What you need to know:

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Solution:

A	$1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \cdots$	$\Leftrightarrow$	$\frac{1}{1}, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \cdots$	$\Leftrightarrow$	$x_n = \frac{1}{n}$
B	$2, 2, 2, 0, 0, 0, 0, 0, \cdots$	$\Leftrightarrow$	$2, 2, 2, 0, 0, 0, 0, 0, \cdots$	$\Leftrightarrow$	$\begin{cases} x_{n\le 3} = 2 \\ x_{n>3} = 0 \end{cases}$
C	$1, -1, 1, -1, \cdots$	$\Leftrightarrow$	$(-1)^2, (-1)^3, (-1)^4, (-1)^5, \cdots$	$\Leftrightarrow$	$x_n = (-1)^{n+1}$
D	$\frac{1}{2}, \frac{3}{4}, \frac{7}{8}, \frac{15}{16}, \cdots$	$\Leftrightarrow$	$1 - \frac{1}{2^1}, 1 - \frac{1}{2^2}, 1 - \frac{1}{2^3}, 1 - \frac{1}{2^4}, \cdots$	$\Leftrightarrow$	$x_n = 1 - \frac{1}{2^n}$
E	$1, 1, 2, 3, 5, 8, 13, \cdots$	$\Leftrightarrow$	$1, 1, 1 + 1, 1 + (1 + 1), (1 + 1) + (1 + (1 + 1)), \cdots$	$\Leftrightarrow$	$\begin{cases} x_1 = 1 = x_2 \\ x_{n>2} = x_{n-2} + x_{n-1} \end{cases}$

Side remark to the sequence D:

Anna-Saray's approach **[closed-form formula]**:

$$\frac{1}{2}, \frac{3}{4}, \frac{7}{8}, \frac{15}{16}, \cdots \Leftrightarrow \frac{2^1 - 1}{2^1}, \frac{2^2 - 1}{2^2}, \frac{2^3 - 1}{2^3}, \frac{2^4 - 1}{2^4}, \cdots \Leftrightarrow x_n = \frac{2^n - 1}{2^n}$$

Finn's approach **[recursive formula]**:

$$\frac{1}{2}, \frac{3}{4}, \frac{7}{8}, \frac{15}{16}, \cdots \Leftrightarrow \frac{1}{2}, \frac{3}{4} = \frac{1 \cdot 2 + 1}{2 \cdot 2} = \frac{1 \cdot 2}{2 \cdot 2} + \frac{1}{2 \cdot 2} = \frac{1}{2} + \frac{1}{2^2}, \frac{7}{8} = \frac{3 \cdot 2 + 1}{4 \cdot 2} = \frac{3}{4} + \frac{1}{2^3}, \frac{15}{16} = \frac{7 \cdot 2 + 1}{8 \cdot 2} = \frac{7}{8} + \frac{1}{2^4}, \cdots \Leftrightarrow \begin{cases} x_1 = \frac{1}{2} \\ x_{n>1} = x_{n-1} + \frac{1}{2^n} \end{cases}$$

P.S.:

You both did a nice job, guys – either approach goes like clockwork

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Finite Series: *task #3*

Task description

Evaluate the following two sums:

A

$$\sum_{i=1}^{3}\left(\sum_{j=1}^{2} y_{ij}\right)^2$$

Given the measurements of y_{ij}:

$i \backslash j$	**1**	**2**	**3**	**4**	**5**
1	1	2	1	3	6
2	3	5	3	1	5
3	4	3	2	5	1
4	6	8	2	3	2

B

$$\sum_{k=2}^{4}\sum_{j=1}^{5}(j+k)$$

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What you need to know:

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Solution:

A

$$\sum_{i=1}^{3}\left(\sum_{j=1}^{2} y_{ij}\right)^2 = \sum_{i=1}^{3}(y_{i1}+y_{i2})^2$$

$$= (y_{11}+y_{12})^2 + (y_{21}+y_{22})^2 + (y_{31}+y_{32})^2$$

$$= (1+2)^2 + (3+5)^2 + (4+3)^2$$

$$= 9 + 64 + 49 = 122$$

i \ j	1	2	3	4	5
1	1	2	1	3	6
2	3	5	3	1	5
3	4	3	2	5	1
4	6	8	2	3	2

B

Basic approach:

$$\sum_{k=2}^{4}\sum_{j=1}^{5}(j+k)=\sum_{k=2}^{4}\left(\sum_{j=1}^{5}(j+k)\right)=\sum_{k=2}^{4}\big((1+k)+(2+k)+(3+k)+(4+k)+(5+k)\big)=$$

$$\sum_{k=2}^{4}(1+k+2+k+3+k+4+k+5+k)=\sum_{k=2}^{4}(1+2+3+4+5+k+k+k+k+k)=$$

$$\sum_{k=2}^{4}\left(\frac{(1_{first}+5_{last})\cdot|\{1,2,3,4,5\}|}{2}+5\cdot k\right)=\sum_{k=2}^{4}\left(\frac{6\cdot 5}{2}+5\cdot k\right)=\sum_{k=2}^{4}(15+5\cdot k)=$$

$$\sum_{k=2}^{4}15+\sum_{k=2}^{4}5\cdot k=15\cdot\sum_{k=2}^{4}1+5\cdot\sum_{k=2}^{4}k=15\cdot(4-2+1)+5\cdot(2+3+4)=15\cdot 3+5\cdot 9=90$$

Alternative approach:

$$\sum_{k=2}^{4}\sum_{j=1}^{5}(j+k)=\sum_{k=2}^{4}\sum_{j=1}^{5}j+\sum_{k=2}^{4}\sum_{j=1}^{5}k=\sum_{k=2}^{4}\sum_{j=1}^{5}j+\sum_{k=2}^{4}\sum_{j=1}^{5}k=\sum_{k=2}^{4}(1+2+3+4+5)+\sum_{k=2}^{4}\left(k\cdot\sum_{j=1}^{5}1\right)$$

$$=\sum_{k=2}^{4}15+\sum_{k=2}^{4}\left(k\cdot\sum_{j=1}^{5}1\right)=15\cdot(4-2+1)+\sum_{k=2}^{4}\big(k\cdot(1+1+1+1+1)\big)$$

$$=15\cdot 3+\sum_{k=2}^{4}k\cdot 5=45+5\cdot\sum_{k=2}^{4}k=45+5\cdot(2+3+4)=45+5\cdot 9=90$$

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Finite Series: *task #4*

Task description

Prove that the following holds true (i.e. for any natural $\boldsymbol{n}$):

$$\sum_{k=1}^{n} (2 \cdot k - 1) = n^2$$

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What you need to know:

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Solution:

The proof is by **induction** on $\boldsymbol{n}$:

First Step [aka Base]

We check whether the statement is true for the initial value of $\boldsymbol{n}$:

$$n = 1$$

The left-hand side [aka **LHS**]:

$$\sum_{k=1}^{n}(2 \cdot k - 1) = \sum_{k=1}^{1}(2 \cdot k - 1) = 2 \cdot 1 - 1 = 1$$

The right-hand side [aka **RHS**]:

$$n^2 = 1^2 = 1$$

As soon as $\mathbf{LHS} = \mathbf{RHS}$ $(1 = 1)$, we are done – let's go ahead and push it forward ☺

Second *(and last)* ***Step*** [aka Inductive]

We assume the statement is true for $\boldsymbol{n-1}$ and check whether it is true for $\boldsymbol{n}$:

(to put in better words, we check if we can climb the ladder for n higher than before)

Assuming the truth for $\boldsymbol{n-1}$ means the following holds true:

$$\sum_{k=1}^{n-1}(2 \cdot k - 1) = (2 \cdot 1 - 1) + (2 \cdot 2 - 1) + (2 \cdot 3 - 1) + \cdots + (2 \cdot (n-1) - 1) =$$

$$1 + 3 + 5 + \cdots + (2 \cdot n - 2 \cdot 1 - 1) = 1 + 3 + 5 + \cdots + (2 \cdot n - 3) = (n-1)^2$$

Then, for one step further, that means for $\boldsymbol{n}$, we have:

$$\sum_{k=1}^{n}(2 \cdot k - 1) = \sum_{k=1}^{n-1}(2 \cdot k - 1) + \sum_{k=n}^{n}(2 \cdot k - 1) = (n-1)^2 + 2 \cdot n - 1 = n^2 - 2 \cdot n + 1 + 2 \cdot n - 1 = n^2$$

Q.E.D (QUOD ERAT DEMONSTRANDUM)

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InFinite Series: *task #5*

Task description

Compute the following sum [aka Guido Grandi's series]:

$$1 - 1 + 1 - 1 + 1 - 1 \cdots$$

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What you need to know:

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Solution:

Firstly, let's try to find a formula for the **sequence of summands** and then use our summation symbol for shortening the visual length of our series:

$$\mathbf{1-1+1-1+\cdots=(-1)^2+(-1)^3+(-1)^4+(-1)^5+\cdots=\overset{\textit{Oh my dayz, what should I put here...☺}}{\sum_{k=1}}(-1)^{k+1}}$$

Alright, let's just do it as it is:

$\mathbf{1-1+1-1++1-1+\cdots=(1-1)+(1-1)+(1-1)+\cdots=0+0+0+\cdots=0}$
Hold on... But what if: $\mathbf{1-1+1-1+1-1+\cdots=1+(-1+1)+(-1+1)+(-1+1)+\cdots=1+0+0+\cdots=1}$
And what if: $\mathbf{1-1+1-1+1-1+\cdots=-1+1-1+1-1+\cdots=-1+(1-1)+(1-1)+\cdots=-1+0+0+\cdots=-1}$
...

Things ain't goin' that well..., right? ☺

Well, the reason is, we need to treat those dots $\cdots$, meaning "**non-stop**" aka "**infinity**" aka "**unbounded**" aka #wasauchimmer☺, in the way that won't mess things up ☺

...

We could talk centuries on which treatment to introduce (in fact, some math fans like Newton, Cesaro, Ramanujan etc. did it for us, centuries ago ☺)

So, to cut things short:

$$\mathbf{1-1+1-1+\cdots}=\sum_{k=1}^{\infty}(-\mathbf{1})^{k+1}=\text{limit of }\sum_{k=1}^{n}(-\mathbf{1})^{k+1}\text{ when } \boldsymbol{n}\text{ gets larger and larger}=\lim_{n\to\infty}\sum_{k=1}^{n}(-\mathbf{1})^{k+1}$$

Observe ***all the outcome possibilities*** for infinite sums (aka series):

have such a **limit** (aka are *convergent*)	**get larger and larger** (aka are ∞-*divergent*)	don't **have** a **limit** (aka are *divergent*)

So, what about our series? – Let's observe the behaviour of its partial sums:

$n = 1$	$\sum_{k=1}^{n}(-1)^{k+1} = \sum_{k=1}^{1}(-1)^{k+1} = (-1)^{1+1} = (-1)^2 = 1$
$n = 2$	$\sum_{k=1}^{n}(-1)^{k+1} = \sum_{k=1}^{2}(-1)^{k+1} = (-1)^{1+1} + (-1)^{2+1} = (-1)^2 + (-1)^3 = 1 + (-1) = 1 - 1 = 0$
$n = 3$	$\sum_{k=1}^{n}(-1)^{k+1} = \sum_{k=1}^{3}(-1)^{k+1} = (-1)^{1+1} + (-1)^{2+1} + (-1)^{3+1} = (-1)^2 + (-1)^3 + (-1)^3 = 1 - 1 + 1 = 1$
…	…

Well, the partial sums give us the following sequence of alternating binaries: $1, 0, 1, 0, 1, 0, \dots$

Wait a second… but how do we know that? – We've checked only 3 partial sums ☺

Question: Does this sequence have some limit value?

Answer: Apparently, not! This sequence just oscillates between 1 and 0, never tending to any limit value at all.

Sounds good ☺, but we ain't done yet ☺ - just one more thing and we'll settle the bill:

One approach [complete the proof as a **homework**]:

$$\sum_{k=1}^{n}(-1)^{k+1} = (-1)^2 \cdot (1 + (-1)^1 + (-1)^2 + \cdots (-1)^{n-1}) = 1 + (-1)^1 + (-1)^2 + \cdots (-1)^{n-1} =$$

$$1 - 1 + (-1)^2 + \cdots (-1)^{n-1} = 0 + (-1)^2 + \cdots (-1)^{n-1} = (-1)^2 + \cdots (-1)^{n-1} = \sum_{k=1}^{n-2}(-1)^{k+1}$$

One more approach [use induction on n and complete the proof as a **homework**]:

$$\sum_{k=1}^{n}(-1)^{k+1} = \frac{1 + (-1)^{n+1}}{2}$$

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Sequence - limits: *task #6*

Task description

Check if $\lim\limits_{n \to \infty} \left(\sqrt{n+1} - \sqrt{n}\right) = 0$

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What you need to know:

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Solution:

Firstly, observe that we are given the sequence $\{x_n\}_{n\in\mathbb{N}}$ defined by the formula $x_n = \sqrt{n+1} - \sqrt{n}$ that allows us to compute each of its terms and also to know its order in the sequence.

So, we are perfectly clear with the task and could start solving it:

$$\lim_{n\to\infty} x_n = \lim_{n\to\infty}\left(\sqrt{n+1} - \sqrt{n}\right) = 0$$

Intuitive (quantitative) definition of a limit:

If this statement is true, $\sqrt{n+1} - \sqrt{n}$ should get closer and closer to $\mathbf{0}$ as n increases – well, if not strictly closer with every larger value of n, then at least closer starting from some n.

$\sqrt{n+1} - \sqrt{n}$ gets closer to $\mathbf{0}$ means that their difference $\left(\sqrt{n+1} - \sqrt{n}\right) - \mathbf{0}$ should get smaller and smaller.

Putting loads of text doesn't make things that clear – so, how to switch to the formal language of mathematics on that?

$\left(\sqrt{n+1} - \sqrt{n}\right) - \mathbf{0}$ gets smaller means its **absolute value** (let's ignore splitting in left/right for a while) should get closer to $\mathbf{0}$ – so, if we take some arbitrarily small positive number and call it ε, then all we say is: $\left|\sqrt{n+1} - \sqrt{n} - 0\right| < \varepsilon$

Let's translate our starting thoughts on limits (up there) into mathematical formalism and we are done:

$\lim_{n\to\infty} x_n = \textit{some number}$ means $|x_n - \textit{some number}| < \varepsilon$ for any $\varepsilon > 0$ once $n > n(\varepsilon) > 0$

Now, we are set to proceed with the solution:

In fact, it is clear even without precise computations that as $\boldsymbol{n}$ goes big, there is no real difference between $\sqrt{\boldsymbol{n}+\mathbf{1}}$ and $\sqrt{\boldsymbol{n}}$ – so, the common sense tells us that the statement is true, – but let's develop the **systematic approach** that helps us to solve any task of this sort:

$$\lim_{n\to\infty} x_n = \textit{some number} \Leftrightarrow \lim_{n\to\infty}\left(\sqrt{n+1}-\sqrt{n}\right) = 0$$

So, our key statement to prove is:

$$|x_n - \textit{some number}| < \varepsilon \Leftrightarrow \left|\left(\sqrt{n+1}-\sqrt{n}\right) - 0\right| < \varepsilon$$

Observe that:

$$\left|\left(\sqrt{n+1}-\sqrt{n}\right)-0\right| = \left|\sqrt{n+1}-\sqrt{n}-0\right| = \left|\sqrt{n+1}-\sqrt{n}\right| = \sqrt{n+1}-\sqrt{n}$$

So, our key statement reduces to:

$$\sqrt{n+1}-\sqrt{n} < \varepsilon$$

We obviously need some "algebraic" trick to solve this for $\boldsymbol{n}$ - the first that comes to mind is:

$$(x-y)\cdot(x+y) = x^2-y^2 \Longrightarrow x-y = \frac{x^2-y^2}{x+y}$$

Applying this identity to our boy ☺ yields:

$$\sqrt{n+1}-\sqrt{n} = \frac{\sqrt{n+1}^2-\sqrt{n}^2}{\sqrt{n+1}+\sqrt{n}} = \frac{n+1-n}{\sqrt{n+1}+\sqrt{n}} = \frac{1}{\sqrt{n+1}+\sqrt{n}} < \varepsilon$$

Things got kinda better, but not quite that much ☺ - so, some smarter trick is needed… what about getting a leaner denominator:

$$\frac{1}{\sqrt{n+1}+\sqrt{n}} < \frac{1}{\sqrt{n}}$$

As soon as:

$$\frac{\mathbf{1}}{\sqrt{\boldsymbol{n}}} < \boldsymbol{\varepsilon} \Longrightarrow \frac{\mathbf{1}}{\sqrt{\boldsymbol{n}+\mathbf{1}}+\sqrt{\boldsymbol{n}}} < \boldsymbol{\varepsilon}$$

We’ve managed to simplify our task even further:

$$\frac{\mathbf{1}}{\sqrt{\boldsymbol{n}}} < \boldsymbol{\varepsilon}$$

Well, this dude gets easily solved:

$$\frac{\mathbf{1}}{\sqrt{\boldsymbol{n}}} < \boldsymbol{\varepsilon} \Leftrightarrow \left(\frac{\mathbf{1}}{\sqrt{\boldsymbol{n}}}\right)^2 < \boldsymbol{\varepsilon}^2 \Leftrightarrow \frac{1}{n} < \boldsymbol{\varepsilon}^2 \Leftrightarrow \boldsymbol{n}^2 \cdot \frac{1}{n} < \boldsymbol{n}^2 \cdot \boldsymbol{\varepsilon}^2 \Leftrightarrow \boldsymbol{n} < \boldsymbol{n}^2 \cdot \boldsymbol{\varepsilon}^2 \Leftrightarrow \mathbf{1} < \boldsymbol{n} \cdot \boldsymbol{\varepsilon}^2 \Leftrightarrow \mathbf{1} \cdot \frac{\mathbf{1}}{\boldsymbol{\varepsilon}^2} < \boldsymbol{n} \cdot \boldsymbol{\varepsilon}^2 \cdot \frac{\mathbf{1}}{\boldsymbol{\varepsilon}^2} \Leftrightarrow \frac{\mathbf{1}}{\boldsymbol{\varepsilon}^2} < \boldsymbol{n} \cdot \mathbf{1} \Leftrightarrow \boldsymbol{n} > \frac{\mathbf{1}}{\boldsymbol{\varepsilon}^2} = \boldsymbol{n}(\boldsymbol{\varepsilon})$$

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Homework Assignments:

are **exam-relevant**, and if completed & submitted/shown prior to the next week class sessions (either in written or oral form), could bring **bonus points** for the exam

Leaderboard: bonus points *per capita* ☺, cumulative

Cheat Sheet of the Day ☺: