GEORG-AUGUST-UNIVERSITÄT
GÖTTINGEN

Mathematics and Computer Science (B.MES.108)
Summer Semester, 2020
Part 1: Linear Algebra for Non-Mathematicians

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$$(AB)^{\top} = B^{\top}A^{\top}$$

$$\mathbb{R}^n \xrightarrow{T} \mathbb{R}^m$$

$$\vec{v} = \sum_{i=1}^{n} \alpha_i \hat{e}_i$$

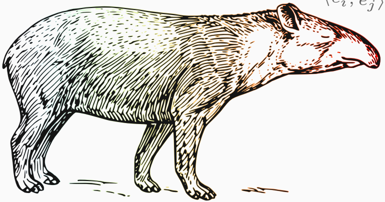
$$A = Q\Lambda Q^{-1}$$

$$\text{Rot}(\theta) = \begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix}$$

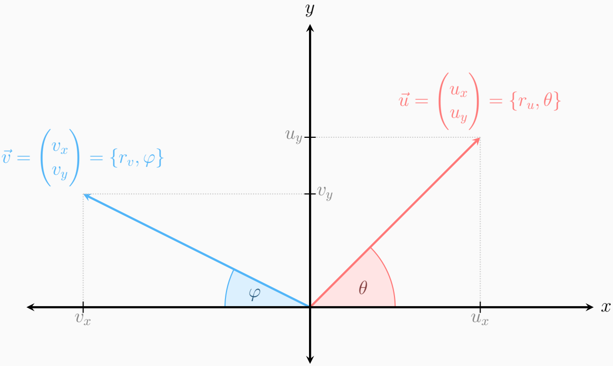
$$A\vec{v} = \lambda\vec{v}$$

$$T(\alpha\vec{u} + \beta\vec{v}) = \alpha T(\vec{u}) + \beta T(\vec{v})$$

$$\langle \hat{e}_i, \hat{e}_j \rangle = \delta_{ij}$$



Chapter 2: Vectors



Basics of Vectors

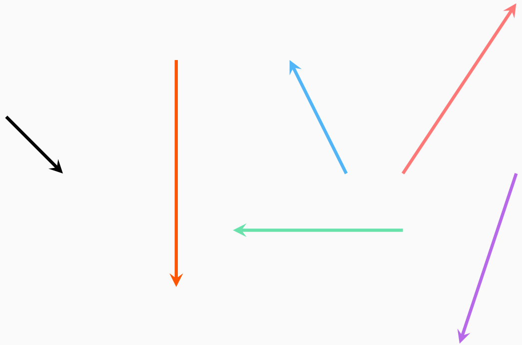
There are 3 distinct approaches to describe what a vector is:

- The physicist's approach (geometric)
- The computer scientist's approach (algebraic)
- The mathematician's approach (abstract)

Geometric Vectors

Definition

A vector is an object with a length and a direction.



Vector Notation

Vectors are denoted as latin letters with an arrow above them:

$$\vec{u}, \quad \vec{v}, \quad \vec{x}, \quad \vec{a}, \quad \cdots$$

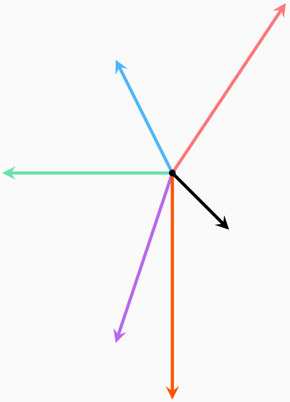
In maths and physics the following notations are mostly used:

$$\boldsymbol{u}, \quad \boldsymbol{v}, \quad \boldsymbol{x}, \quad \boldsymbol{a}, \quad \cdots$$

$$\underline{\boldsymbol{u}}, \quad \underline{\boldsymbol{v}}, \quad \underline{\boldsymbol{x}}, \quad \underline{\boldsymbol{a}}, \quad \cdots$$

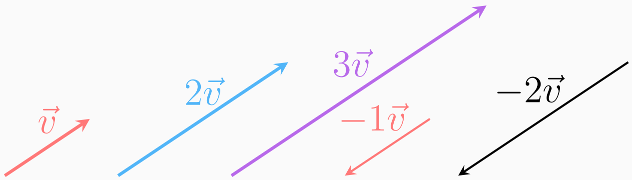
Geometric Vectors

We consider all vectors starting at the same point, called the **origin**.



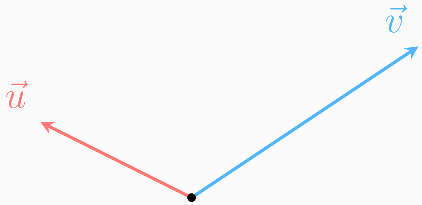
Scaling Vectors

We can multiply a vector by a real number, which we refer to as a **scalar** . This scales only the length of the vector while keeping its direction on the same line as before:



Vector Addition

Adding two vectors is done by placing the origin of one vector at the head of the other vector. The addition results in a vector starting at the first vector's origin and ending at the second vector's head:



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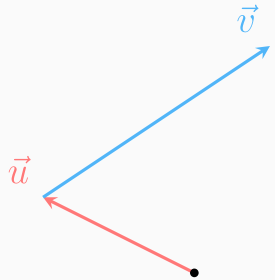

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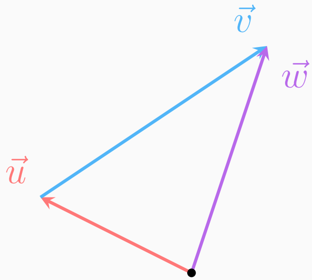

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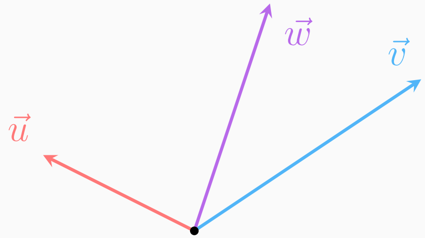
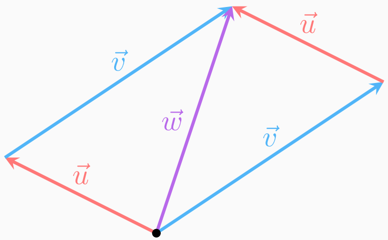


Vector Addition

Notice that adding vectors is a commutative operation, i.e.

$$\vec{u} + \vec{v} = \vec{v} + \vec{u}$$



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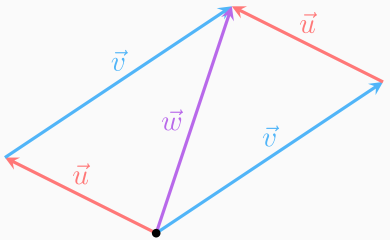

This is refered to as the **parallelogram law of vector addition**.

The Zero Vector

And important vector is the **zero vector**, which has a length of 0 and no direction. It is notated as $\vec{0}$, and is neutral to addition, i.e. for any vector $\vec{v}$:

$$\vec{v} + \vec{0} = \vec{0} + \vec{v} = \vec{v}.$$

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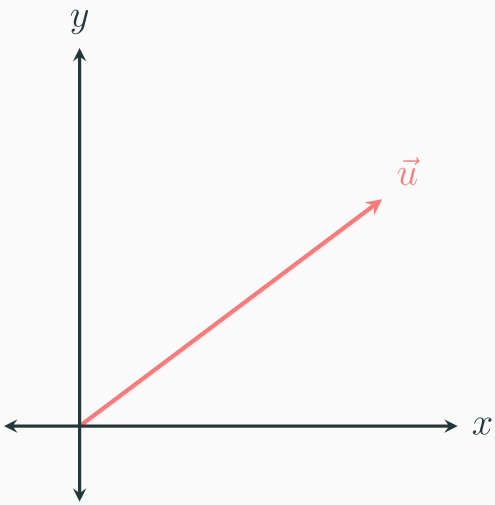
$$\vec{v} + \vec{0} = \vec{0} + \vec{v} = \vec{v}.$$

Similarily, any addition of a vector with its opposite vector results in the zero vector:

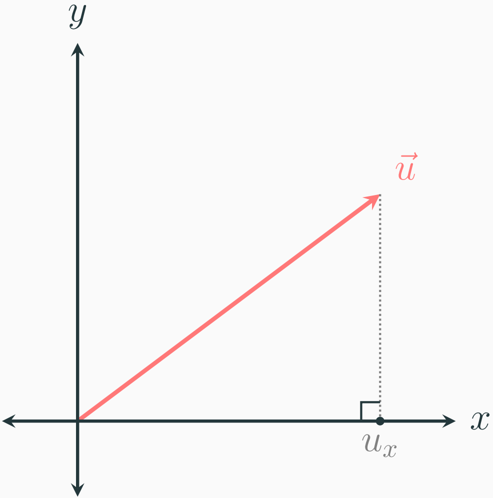
$$\vec{v} + (-\vec{v}) = -\vec{v} + \vec{v} = \vec{0}.$$

Placing a vector in a cartesian coordinate system:

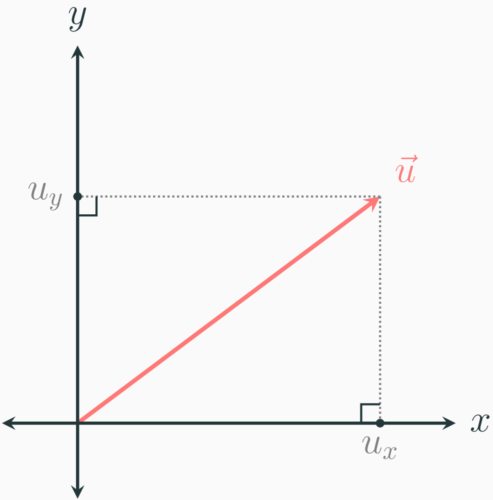


Algebraic Vectors

Then, drawing a perpendicular from $\vec{v}$ to the x-axis:

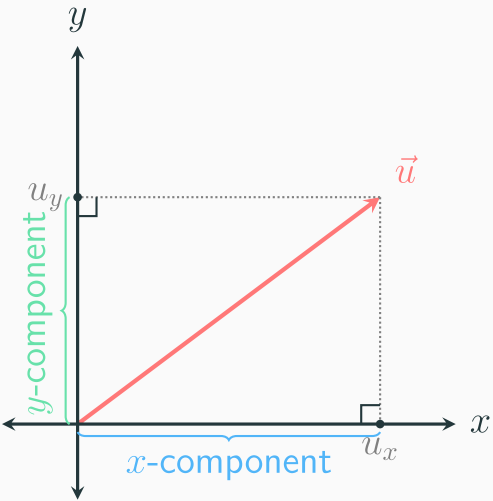


And similarily for the y-axis:



Algebraic Vectors

We call u_x and u_y the **components** of $\vec{u}$.



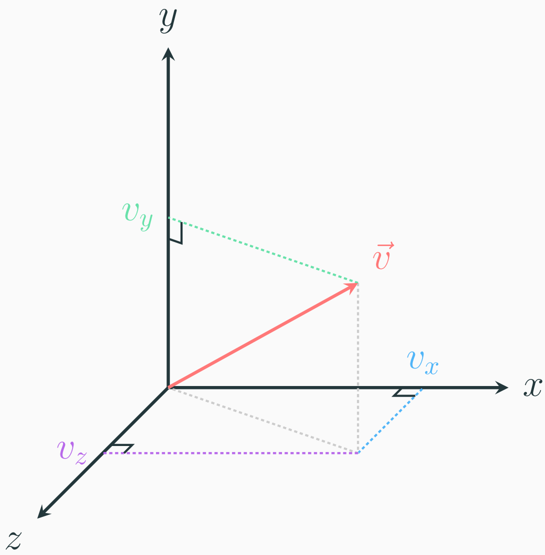
Column Vectors

We then notate the vector $\vec{u}$ as a **column vector** with components u_x, u_y:

$$\vec{u} = \begin{pmatrix} u_x \\ u_y \end{pmatrix}.$$

Since $\vec{u}$ has two real components, it is a member of $\mathbb{R}^2$.

This scheme can be extended to 3-dimensional vectors:



Higher-dimensional Vectors

A column vector in $\mathbb{R}^3$ looks as following:

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$$\vec{v} = \begin{pmatrix} v_x \\ v_y \\ v_z \end{pmatrix},$$

and in $\mathbb{R}^4$:

$$\vec{a} = \begin{pmatrix} v_x \\ v_y \\ v_z \\ v_w \end{pmatrix}.$$

Higher-dimensional Vectors

A general column vector in $\mathbb{R}^n$ looks as following:

$$\vec{v} = \begin{pmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{pmatrix} \left.\vphantom{\begin{pmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{pmatrix}}\right\} n \text{ components}$$

The Zero Vector

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In $\mathbb{R}^3$ it is

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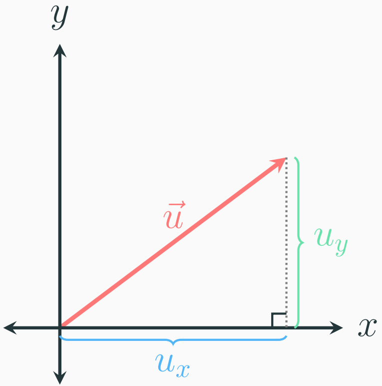
And generally, in $\mathbb{R}^n$, it is

$$\vec{0} = \left.\begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}\right\} n \text{ components}$$

Length and Angle of a Vector

Using the Pythagorean theorem to calculate the length (norm) of a vector in $\mathbb{R}^2$:

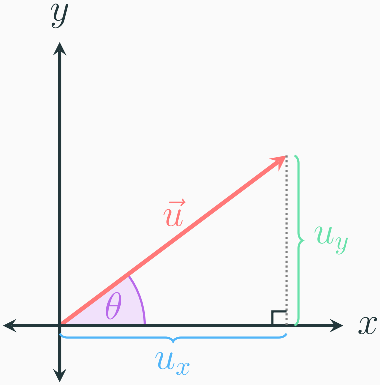
$$\|\vec{u}\| = \sqrt{u_x^2 + u_y^2}.$$



Length and Angle of a Vector

The angle θ is then:

$$\tan(\theta) = \frac{u_y}{u_x}.$$



Length of a Vector

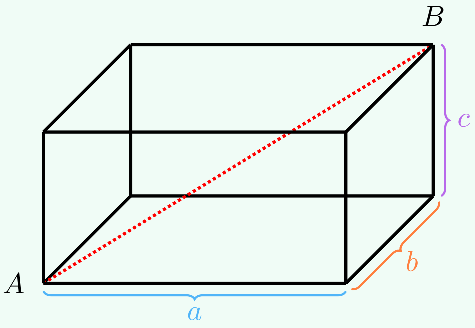
Similarly, the length of a column vector in $\mathbb{R}^3$, $\vec{v} = \begin{pmatrix} v_x \\ v_y \\ v_z \end{pmatrix}$ is

$$\|\vec{v}\| = \sqrt{v_x^2 + v_y^2 + v_z^2}.$$

Length of a Vector

Challenge

Show that the above given formula is true, i.e. show that for a box of sides a, b, c, the length of the line from A to B (see figure) is indeed $\sqrt{a^2 + b^2 + c^2}$.



Length of a Vector

For a general n-dimensional vector $\vec{w} = \begin{pmatrix} w_1 \\ w_2 \\ \vdots \\ w_n \end{pmatrix}$,

$$\begin{aligned} \|\vec{w}\| &= \sqrt{w_1^2 + w_2^2 + \cdots + w_n^2} \\ &= \sqrt{\sum_{i=1}^{n} w_i^2}. \end{aligned}$$

Scaling Vectors

Scaling a column vector $\vec{v}$ by a scalar α is done by multiplying each of its components by α:

$$\vec{v} = \begin{pmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{pmatrix} \quad \Rightarrow \quad \alpha\vec{v} = \begin{pmatrix} \alpha v_1 \\ \alpha v_2 \\ \vdots \\ \alpha v_n \end{pmatrix}.$$

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Example

$$\vec{a} = \begin{pmatrix} 1 \\ -2 \\ 7 \end{pmatrix} \quad \Rightarrow \quad 5\vec{a} = \begin{pmatrix} 5 \\ -10 \\ 35 \end{pmatrix}.$$

Scaling Vectors

Proof

The length of $\alpha\vec{v} = \begin{pmatrix} \alpha v_1 \\ \alpha v_2 \\ \vdots \\ \alpha v_n \end{pmatrix}$ is

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Normalizing Vectors

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Normalization of a vector is an operation that scales the vector to be of length 1 without changing its direction.

It is done by scaling the vector by the reciprocal of its norm. We notate the result by a "hat" symbol:

$$\hat{v} = \frac{1}{\|\vec{v}\|}\vec{v}.$$

Normalizing Vectors

Example

For $\vec{w} = \begin{pmatrix} -3 \\ 4 \end{pmatrix}$,

$$\|\vec{w}\| = \sqrt{(-3)^2 + 4^2} = \sqrt{9 + 16} = \sqrt{25} = 5.$$

Normalizing Vectors

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$$\|\vec{w}\| = \sqrt{(-3)^2 + 4^2} = \sqrt{9 + 16} = \sqrt{25} = 5.$$

Thus,

$$\hat{w} = \frac{1}{\|\vec{w}\|}\vec{w} = \frac{1}{5}\begin{pmatrix} -3 \\ 4 \end{pmatrix} = \begin{pmatrix} -\frac{3}{5} \\ \frac{4}{5} \end{pmatrix} = \begin{pmatrix} -0.6 \\ 0.8 \end{pmatrix}.$$

Normalizing Vectors

Challenge

Show that dividing any vector

$$\vec{v} = \begin{pmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{pmatrix}$$

by its norm always results in a vector of the same direction and a norm of 1.

Vector Addition

Addition of two column vectors is done **component-wise**, i.e.

$$\begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{pmatrix} + \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{pmatrix} = \begin{pmatrix} a_1 + b_1 \\ a_2 + b_2 \\ \vdots \\ a_n + b_n \end{pmatrix}.$$

Vector Addition

Example

$$\begin{pmatrix} 3 \\ -5 \end{pmatrix} + \begin{pmatrix} 2 \\ 0 \end{pmatrix} = \begin{pmatrix} 5 \\ -5 \end{pmatrix}, \quad \begin{pmatrix} -7 \\ 2 \end{pmatrix} + \begin{pmatrix} 1 \\ 0.5 \end{pmatrix} = \begin{pmatrix} -6 \\ 2.5 \end{pmatrix},$$

$$\begin{pmatrix} -1 \\ 0 \\ 2 \end{pmatrix} + \begin{pmatrix} 1 \\ 0 \\ -2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}, \quad \begin{pmatrix} 5 \\ 0.5 \\ -1 \end{pmatrix} + \begin{pmatrix} -5 \\ 0.5 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}.$$

Vector Addition

Subtraction of two vectors $\vec{u}$ and $\vec{v}$ is equivalent to the addition

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Example

$$\begin{pmatrix} 3 \\ -1 \end{pmatrix} - \begin{pmatrix} 5 \\ 2 \end{pmatrix} = \begin{pmatrix} 3 \\ -1 \end{pmatrix} + \begin{pmatrix} -5 \\ -2 \end{pmatrix} = \begin{pmatrix} -2 \\ -3 \end{pmatrix} .$$

Vector Addition

Note

Addition of two vectors of different dimensionality (e.g. $\mathbb{R}^2$ and $\mathbb{R}^3$) is **undefined**.

Linear Combination of Vectors

A **linear combination** of two vectors $\vec{u}, \vec{v}$ is an expression of the form

$$\alpha\vec{u} + \beta\vec{v},$$

where $\alpha, \beta \in \mathbb{R}$.

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Example

A linear combination of the vectors $\vec{u} = \begin{pmatrix} 2 \\ -12 \end{pmatrix}$, $\vec{v} = \begin{pmatrix} 0 \\ 3 \end{pmatrix}$:

$$0.5\vec{u} + 2\vec{v} = \begin{pmatrix} 1 \\ -6 \end{pmatrix} + \begin{pmatrix} 0 \\ 6 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$

Linear Combination of Vectors

The definition can be extended to any $n \in \mathbb{N}$ vectors:

$$\alpha_1 \vec{v}_1 + \alpha_2 \vec{v}_2 + \cdots + \alpha_n \vec{v}_n = \sum_{i=1}^{n} \alpha_i \vec{v}_i.$$

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Example

A linear combination of four vectors in $\mathbb{R}^3$:

$$\begin{pmatrix} 1 \\ 4 \\ 0 \end{pmatrix} + 3\begin{pmatrix} 0 \\ -1 \\ 5 \end{pmatrix} - 7\begin{pmatrix} -2 \\ 1 \\ 2 \end{pmatrix} + 0.5\begin{pmatrix} 6 \\ 4 \\ 2 \end{pmatrix} = \begin{pmatrix} 18 \\ -4 \\ 2 \end{pmatrix}.$$

Linear Combination of Vectors

Note

Note that the result of a linear combination of vectors is always a vector.

Linear (In)Dependence of Vectors

Two vectors $\vec{u}$ and $\vec{v}$ are **linearly dependent** if one of them is a scale of the other, i.e. if

$$\vec{u} = \alpha\vec{v} \text{ or } \vec{v} = \beta\vec{u}.$$

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Example

Examples of sets of two linearly dependent vectors:

$$\left\{ \begin{pmatrix} 1 \\ -3 \end{pmatrix}, \begin{pmatrix} 2 \\ -6 \end{pmatrix} \right\} \quad \left\{ \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -3 \\ -3 \\ 0 \end{pmatrix} \right\}$$

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Example

Examples of sets of two linearly dependent vectors:

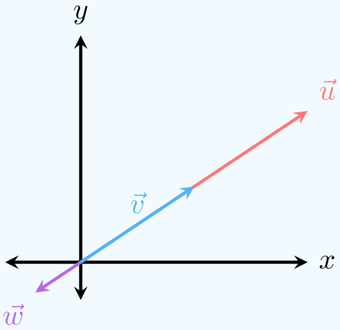
$$\left\{\begin{pmatrix}-2\\1\\4\end{pmatrix}, \begin{pmatrix}1\\-0.5\\2\end{pmatrix}\right\} \quad \left\{\begin{pmatrix}1\\-2\\5\\-3\end{pmatrix}, \begin{pmatrix}3\\-6\\15\\-9\end{pmatrix}\right\}$$

Linear (In)Dependence of Vectors

The geometric interpretation of two linearly dependent vectors is that they lie on the same line in space.

Example

The following vectors all lie on the same line in $\mathbb{R}^2$:



The definition of linear dependence can be extended to any number $n \in \mathbb{N}$ of vectors:

Definition

A set of vectors $\{\vec{v}_1, \vec{v}_2, \cdots, \vec{v}_n\}$ is **linearly dependent** if there exists a set of coefficients $\{\alpha_1, \alpha_2, \cdots, \alpha_n\}$, **not all of them 0**, such that

$$\sum_{i=1}^{n} \alpha_i \vec{v}_i = \alpha_1 \vec{v}_1 + \alpha_2 \vec{v}_2 + \cdots + \alpha_n \vec{v}_n = \vec{0}.$$

The definition is equivalent to having at least one vector in the set which is a linear combination of the other vectors in the set.

Example

The following vectors in $\mathbb{R}^3$ form a linearly dependent set:

$$\vec{u} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}, \quad \vec{v} = \begin{pmatrix} -1 \\ 6 \\ 1 \end{pmatrix}, \quad \vec{w} = \begin{pmatrix} 2 \\ 0 \\ 4 \end{pmatrix},$$

since $\vec{v} = 3\vec{u} - 2\vec{w}$.

Linear (In)Dependence of Vectors

Another equivalent definition is that of a **linearly independent** set of vectors:

Definition

A set of vectors $\{\vec{v}_1, \vec{v}_2, \cdots, \vec{v}_n\}$ is **linearly independent** if the equation

$$\sum_{i=1}^{n} \alpha_i \vec{v}_i = \alpha_1 \vec{v}_1 + \alpha_2 \vec{v}_2 + \cdots + \alpha_n \vec{v}_n = \vec{0}$$

is only true when $\alpha_1 = \alpha_2 = \cdots = \alpha_n = 0$ (i.e. if all the coefficients are equal to zero, also known as the **trivial solution**).

Spaces, Subspaces and Basis Sets

Any vector in $\mathbb{R}^2$ can be constructed from a linear combination of two **linearly independent** 2-dimensional vectors.

Spaces, Subspaces and Basis Sets

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Example

Using the vectors $\vec{u} = \begin{pmatrix} 1 \\ 3 \end{pmatrix}$, $\vec{v} = \begin{pmatrix} 0 \\ -2 \end{pmatrix}$:

$$\begin{pmatrix} 2 \\ 0 \end{pmatrix} = 2\vec{u}+3\vec{v}, \quad \begin{pmatrix} -1 \\ -11 \end{pmatrix} = -\vec{u}+4\vec{v}, \quad \begin{pmatrix} -2 \\ 10 \end{pmatrix} = -2\vec{u}-8\vec{v}.$$

Any vector in $\mathbb{R}^2$ can be constructed from a linear combination of two **linearly independent** 2-dimensional vectors.

Example

Using the vectors $\vec{u} = \begin{pmatrix} 1 \\ 3 \end{pmatrix}$, $\vec{v} = \begin{pmatrix} 0 \\ -2 \end{pmatrix}$:

$$\begin{pmatrix} 2 \\ 0 \end{pmatrix} = 2\vec{u}+3\vec{v}, \quad \begin{pmatrix} -1 \\ -11 \end{pmatrix} = -\vec{u}+4\vec{v}, \quad \begin{pmatrix} -2 \\ 10 \end{pmatrix} = -2\vec{u}-8\vec{v}.$$

Generally:

$$\begin{pmatrix} a \\ b \end{pmatrix} = a\vec{u} + \frac{3a-b}{2}\vec{v}.$$

Note

The reason why any two linearly independent vectors in $\mathbb{R}^2$, $\vec{u}, \vec{v}$, span all of $\mathbb{R}^2$, i.e. that any vector $\vec{w} = \begin{pmatrix} w_x \\ w_y \end{pmatrix}$ can be expressed as a linear combination of $\vec{u}$ and $\vec{v}$, is that the linear system

$$\begin{cases} \alpha u_x & + & \beta v_x & = & w_x \\ \alpha u_y & + & \beta v_y & = & w_y \end{cases}$$

always has a solution under the conditions forced by the linear independence of $\vec{u}$ and $\vec{v}$. Linear systems will be discussed in Chapter 5 (Systems of Linear Equations).

Spaces

As with two linearly independent vectors in $\mathbb{R}^2$, any three linearly independent vectors in $\mathbb{R}^3$ span all of $\mathbb{R}^3$.

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Generally, any set of $n \in \mathbb{N}$ linearly independent vectors in $\mathbb{R}^n$ span all of $\mathbb{R}^n$, i.e. any vector in $\mathbb{R}^n$ can be expressed as a linear combination of a set of $n \in \mathbb{N}$ linearly independent vectors in $\mathbb{R}^n$.

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Generally, any set of $n \in \mathbb{N}$ linearly independent vectors in $\mathbb{R}^n$ span all of $\mathbb{R}^n$, i.e. any vector in $\mathbb{R}^n$ can be expressed as a linear combination of a set of $n \in \mathbb{N}$ linearly independent vectors in $\mathbb{R}^n$.

We call such a set a **basis set** of $\mathbb{R}^n$.

Basis Sets

Example

In $\mathbb{R}^2$, the following sets of two vectors are all linearly independent, and thus are basis sets of $\mathbb{R}^2$:

$$\left\{ \begin{pmatrix} 1 \\ -2 \end{pmatrix}, \begin{pmatrix} 0 \\ 3 \end{pmatrix} \right\} \quad \left\{ \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ -1 \end{pmatrix} \right\} \quad \left\{ \begin{pmatrix} 4 \\ -1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \end{pmatrix} \right\}$$

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And similarily for $\mathbb{R}^3$:

$$\left\{ \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ -1 \\ 2 \end{pmatrix} \right\} \quad \left\{ \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 2 \end{pmatrix}, \begin{pmatrix} 2 \\ 3 \\ 0 \end{pmatrix} \right\}$$

Basis Sets

If all the vectors of a basis set are orhtogonal to each other, then the set is called an **orthogonal basis set** [1].

[1] Orthogonality is a generalization of perpendicularity, i.e. having a right angle, for any abstract space. In this course we use the term **orthogonal** instead of **perpendicular**.

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If all the vectors of a basis set are orhtogonal to each other, then the set is called an **orthogonal basis set** [1].

If in addition to being orthogonal, all the vectors are also normalized, then the set is an **orthonormal basis set**.

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Example

In $\mathbb{R}^2$ the following set is an orthogonal set:

$$\left\{ \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} -1 \\ 1 \end{pmatrix} \right\}$$

since the angle between $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ and the x-axis is $\theta_1 = \arctan\left(\frac{1}{1}\right) = 45°$, the angle between $\begin{pmatrix} -1 \\ 1 \end{pmatrix}$ and the x-axis is $\theta_2 = \arctan\left(\frac{1}{-1}\right) = 135°$, and the difference between these angles is $\theta_2 - \theta_1 = 90°$.

Example

If we take the above set and normalize each vector (the normalization factor for both is $\frac{1}{\sqrt{2}}$), we get an orthonormal basis set:

$$\left\{ \begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix}, \begin{pmatrix} -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix} \right\}$$

Basis Sets

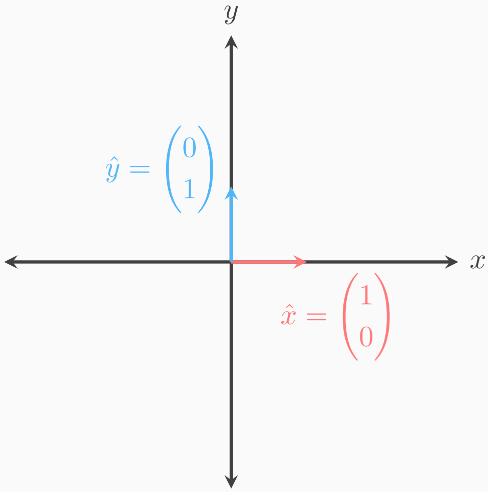
In $\mathbb{R}^2$ the basis

$$\left\{ \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\},$$

which is an orthonormal set, is known as the **standard basis**.

The vectors $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$ are denoted as $\hat{x}$ and $\hat{y}$, respectively.

Basis Sets



y
$\hat{y} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$
x
$\hat{x} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

Basis Sets

Similarily, in $\mathbb{R}^3$ the standard basis is

$$\left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right\},$$

with the vectors also named $\hat{x}, \hat{y}$ and $\hat{z}$, respectively.

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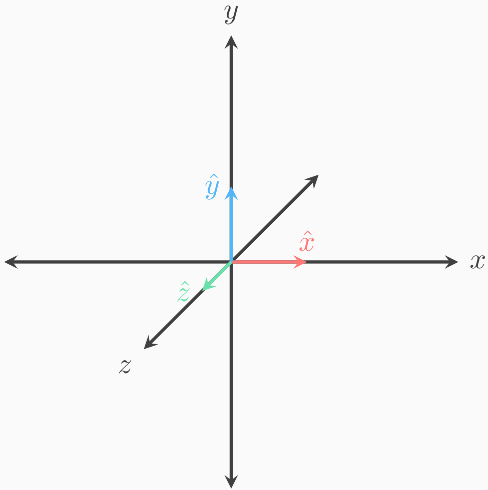
Note

On both $\mathbb{R}^2$ and $\mathbb{R}^3$, $\hat{x}$ and $\hat{y}$ are also sometimes called $\hat{i}$ and $\hat{j}$, respectively, while $\hat{z}$ in $\mathbb{R}^3$ is also called $\hat{k}$.

Basis Sets



Basis Sets

In general, the standard basis set in $\mathbb{R}^n$ is the set of vectors

$$\left\{ \hat{e}_1 = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix},\ \hat{e}_2 = \begin{pmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{pmatrix},\ \cdots,\ \hat{e}_n = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{pmatrix} \right\},$$

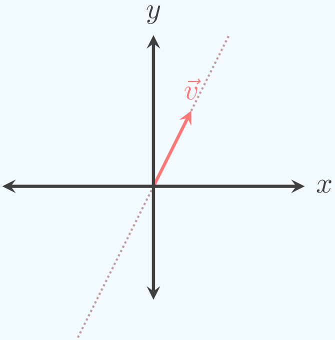
i.e. where the i-th basis vector is a vector that has 1 as its i-th component, and the rest of the components are 0.

Subspaces

In $\mathbb{R}^2$ every non-zero vector spans a line in $\mathbb{R}^2$, **going through the origin**. We call this line a **subspace** of $\mathbb{R}^2$.

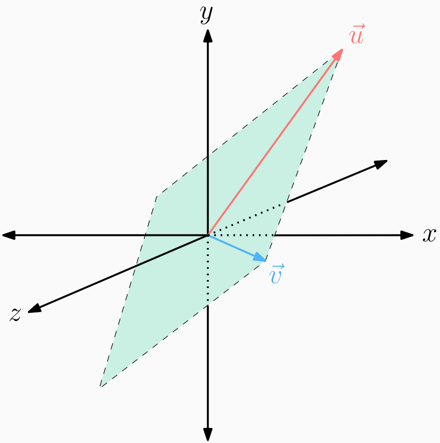
Example

The vector $\vec{u} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$ spans a line of slope $m = 3$ going through the origin. Any vector that is a scale of $\vec{u}$ lies on this line, and is in this subspace.



Subspaces

Similarily, any non-zero vector in $\mathbb{R}^3$ also spans a line going through the origin. In addition, any two linearly independent vectors span a **plane** going through the origin.

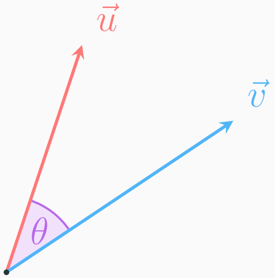


Subspaces

And generally, any set of $m < n$ linearly independent vectors in $\mathbb{R}^n$ span a subspace of $\mathbb{R}^n$ which goes through the origin.

The Dot Product

As discussed, any two linearly independent vectors $\vec{u}, \vec{v} \in \mathbb{R}^n$ span a plane which goes through the origin of $\mathbb{R}^n$. In that plane, there is some angle θ between the vectors.



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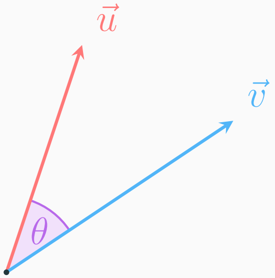


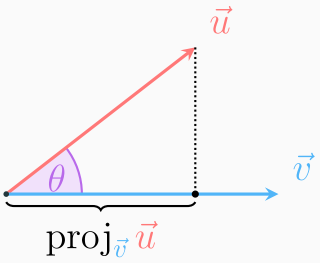
How can we calculate θ?

The Dot Product

If we rotate the two vectors such that one of them lies on the horizontal direction, we can draw a perpendicular line from $\vec{u}$ to $\vec{v}$. Using trigonometry we get

$$\cos(\theta) = \frac{\operatorname{proj}_{\vec{v}} \vec{u}}{\|\vec{u}\|},$$

where $\operatorname{proj}_{\vec{v}} \vec{u}$ is the length of the projection of $\vec{u}$ on $\vec{v}$.



The Dot Product

We define the magnitude $\operatorname{proj}_{\vec{v}} \vec{u} \cdot \|\vec{u}\|$ (i.e. the length of the projection of $\vec{u}$ on $\vec{v}$ multiplied by the length of $\vec{v}$) as the **dot product** of $\vec{u}$ and $\vec{v}$.

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Two common notations for the dot product of two vectors $\vec{a}, \vec{b}$ are

1. $\vec{a} \cdot \vec{b}$ (the one used in this course), and
2. $\langle \vec{a}, \vec{b} \rangle$.

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1. $\vec{a} \cdot \vec{b}$ (the one used in this course), and
2. $\langle \vec{a}, \vec{b} \rangle$.

A more common formulation of the dot product is

$$\vec{u} \cdot \vec{v} = \|\vec{u}\| \, \|\vec{v}\| \cos(\theta).$$

The Dot Product

Some properties of the dot product:

- It is non-negative, i.e. $\vec{u} \cdot \vec{v} \geq 0$.

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- It equals zero in only one of two cases:
  1. One of the vectors (or both) is the zero vector, or
  2. The angle θ between the vectors is 90° (since then $\cos(\theta) = \cos(90°) = 0$).

The Last point is so important that it's worth framing it and hanging it on a wall[2]. We will forgoe the hanging part here, and only frame it:

[2]Preferably, above your bed so you see it when you wake up and when you go to sleep.

The Dot Product

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Note

When the dot product of two (non zero) vectors is equal to zero, they are orthogonal to each other.

$$\Updownarrow$$

When two (non zero) vectors are orthogonal to each other, their dot product is zero.

[2]Preferably, above your bed so you see it when you wake up and when you go to sleep.

The Dot Product

Example

What is the dot product of the two vectors $\vec{v} = \begin{pmatrix} 4 \\ 4 \end{pmatrix}$ and $\vec{v} = \begin{pmatrix} -1 \\ 2 \end{pmatrix}$?

The angle θ between $\vec{u}$ and the x-axis is

$$\tan(\theta) = \frac{4}{4} = 1 \quad \Rightarrow \quad \theta = 45^\circ.$$

The angle φ between $\vec{v}$ and the x-axis is

$$\tan(\varphi) = \frac{2}{-1} = -2 \quad \Rightarrow \quad \varphi \approx 116.57^\circ.$$

Example

Thus, the angle between the two vectors is $\omega = \varphi - \theta =$ 71.57°.

The norm of $\vec{u}$ is

$$\|\vec{u}\| = \sqrt{4^2 + 4^2} = \sqrt{16 + 16} = \sqrt{32},$$

and of $\vec{v}$ is

$$\|\vec{v}\| = \sqrt{(-1)^2 + 2^2} = \sqrt{1 + 4} = \sqrt{5}.$$

Thus, the dot product of the two vectors is:

$$\vec{u} \cdot \vec{v} = \sqrt{32}\sqrt{5}\cos(71.57^\circ) = \sqrt{160} \cdot 0.32 = 4.$$

The Dot Product

When two vectors are given as column vectors, their dot product can be calculated as the sum of their component-wise product, i.e.

$$\begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{pmatrix} \cdot \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{pmatrix} = a_1 b_1 + a_2 b_2 + \cdots + a_n b_n = \sum_{i=1}^{n} a_i b_i .$$

The Dot Product

Example

Using the vectors $\vec{u} = \begin{pmatrix} 4 \\ 4 \end{pmatrix}$ and $\vec{v} = \begin{pmatrix} -1 \\ 2 \end{pmatrix}$ from the previous example, we get

$$\vec{u} \cdot \vec{v} = 4 \cdot (-1) + 4 \cdot 2 = -4 + 8 = 4,$$

which is exactly the result we got in the previous example.

The Cross Product

Another product of two vectors is the **cross product**. Unlike the dot product, the cross product is only defined on $\mathbb{R}^3$ (and with a somewhat different meaning on $\mathbb{R}^2$ as well).

The Cross Product

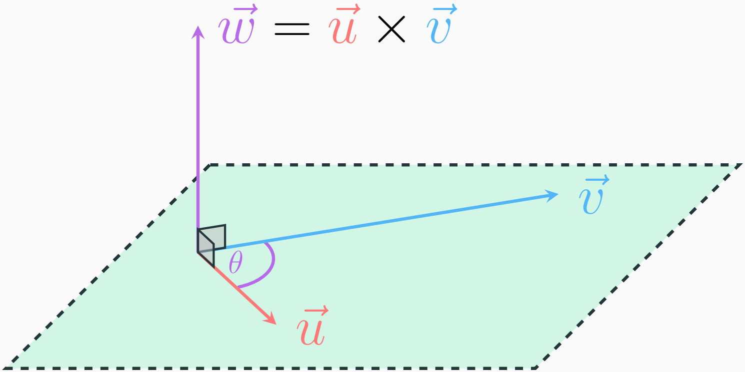
Another product of two vectors is the **cross product**. Unlike the dot product, the cross product is only defined on $\mathbb{R}^3$ (and with a somewhat different meaning on $\mathbb{R}^2$ as well).

Geometrically, the cross product of two vectors $\vec{u}, \vec{v} \in \mathbb{R}^3$ is defined as a vector $\vec{w}$ which is **orthogonal to both** $\vec{u}$ and $\vec{v}$, and has a magnitude

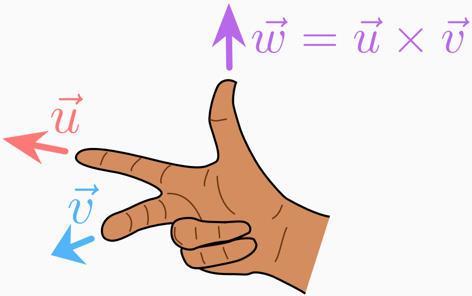
$$r_w = \|\vec{u}\| \, \|\vec{v}\| \sin(\theta),$$

where θ is the angle between $\vec{u}$ and $\vec{v}$.

The Cross Product



The direction of $\vec{u} \times \vec{v}$ is determined by the **right-hand rule**: using a person's right hand, when $\vec{u}$ points in the direction of their index finger and $\vec{v}$ in the direction of their middle finger, then $\vec{w} = \vec{u} \times \vec{v}$ points in the direction of their thumb:



The Cross Product

The cross product is **anti-commutative**, i.e. changing the order of the vectors results in inverting the product:

$$\vec{u} \times \vec{v} = -\left(\vec{v} \times \vec{u}\right).$$

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When the vectors are given as column vectors $\vec{u} = \begin{pmatrix} u_x \\ u_y \\ u_z \end{pmatrix}$, $\vec{v} = \begin{pmatrix} v_x \\ v_y \\ v_z \end{pmatrix}$, the resulting cross product is

$$\vec{u} \times \vec{v} = \begin{pmatrix} u_y v_z - u_z v_y \\ u_z v_x - u_x v_z \\ u_x v_y - u_y v_x \end{pmatrix}$$

The Cross Product

Example

What is the cross product of $\hat{e}_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$ and $\hat{e}_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$?

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$$\hat{e}_1 \times \hat{e}_2 = \begin{pmatrix} \cancel{0 \cdot 0} - \cancel{0 \cdot 1} \\ \cancel{0 \cdot 0} - \cancel{1 \cdot 0} \\ 1 \cdot 1 - \cancel{0 \cdot 0} \end{pmatrix}$$

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The Cross Product

Note

The cross product of two of the standard basis vectors in $\mathbb{R}^3$ is the third basis vector. Its sign ($\pm$) is determined by a cyclic rule:

$$\text{sign}\left(\hat{e}_i \times \hat{e}_j\right) = \begin{cases} 1 & \text{if } (i,j) \in \left\{(1,2),\ (2,3),\ (3,1)\right\}, \\ -1 & \text{if } (i,j) \in \left\{(3,2),\ (2,1),\ (1,3)\right\}, \\ 0 & \text{otherwise.} \end{cases}$$

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Challenge

Using component calculation and utilizing the dot product, show that $\vec{u} \times \vec{v}$ is indeed orthogonal to both $\vec{u}$ and $\vec{v}$.