GEORG-AUGUST-UNIVERSITÄT
GÖTTINGEN

Mathematics and Computer Science (B.MES.108)
Summer Semester, 2020
Part 1: Linear Algebra for Non-Mathematicians

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$(AB)^{\top} = B^{\top}A^{\top}$

$\mathbb{R}^n \xrightarrow{T} \mathbb{R}^m$

$\vec{v} = \sum_{i=1}^{n} \alpha_i \hat{e}_i$

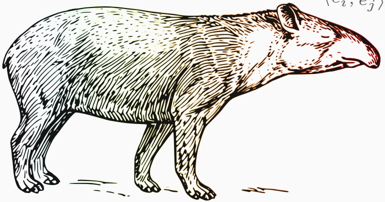
$A = Q\Lambda Q^{-1}$

$\text{Rot}(\theta) = \begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix}$

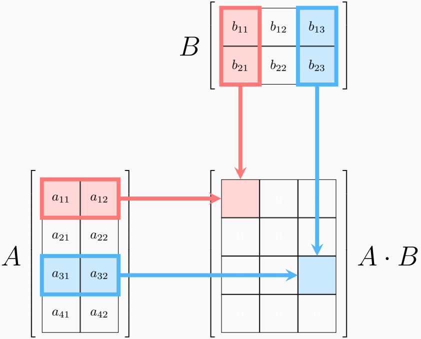
$A\vec{v} = \lambda\vec{v}$

$T(\alpha\vec{u} + \beta\vec{v}) = \alpha T(\vec{u}) + \beta T(\vec{v})$

$\langle \hat{e}_i, \hat{e}_j \rangle = \delta_{ij}$



Chapter 4: Matrices



Matrices from Linear Transformations

Any vector in a space $\mathbb{R}^n$ can be written as a linear combination of the elements of a base in $\mathbb{R}^n$.

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The vector $\begin{pmatrix} 1 \\ -3 \end{pmatrix}$ can be written as a linear combination of the standard basis vectors in $\mathbb{R}^2$:

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$$\begin{aligned} \vec{v} &= \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ -3 \end{pmatrix} \\ &= \begin{pmatrix} 1 \\ 0 \end{pmatrix} - 3 \begin{pmatrix} 0 \\ 1 \end{pmatrix} \end{aligned}$$

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Matrices from Linear Transformations

When a linear transformation T is applied to a vector $\vec{v} = \alpha_1\hat{e}_1 + \alpha_2\hat{e}_2 + \cdots + \alpha_n\hat{e}_n$, linearity dictates the following:

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$$
\begin{aligned}
T(\vec{v}) &= T(\alpha_1 \hat{e}_1 + \alpha_2 \hat{e}_2 + \cdots + \alpha_n \hat{e}_n) \\
&= T(\alpha_1 \hat{e}_1) + T(\alpha_2 \hat{e}_2) + \cdots + T(\alpha_n \hat{e}_n)
\end{aligned}
$$

Additivity

Matrices from Linear Transformations

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T(\vec{v}) &= T(\alpha_1 \hat{e}_1 + \alpha_2 \hat{e}_2 + \cdots + \alpha_n \hat{e}_n) \\
\text{Additivity} \rightarrow &= T(\alpha_1 \hat{e}_1) + T(\alpha_2 \hat{e}_2) + \cdots + T(\alpha_n \hat{e}_n) \\
\text{Scalability} \rightarrow &= \alpha_1 T(\hat{e}_1) + \alpha_2 T(\hat{e}_2) + \cdots + \alpha_n T(\hat{e}_n),
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i.e. - the transformed vector is a linear combination of the transformed standard basis vectors, with the components of the original vector as coefficients.

Example

Consider the vector $\vec{v} = \begin{pmatrix} 3 \\ -1 \end{pmatrix}$, and the transformation $T\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -2x + y \\ 3x - 2y \end{pmatrix}$. We can calculate the transformation $T(\vec{v})$ directly:

$$T(\vec{v}) = \begin{pmatrix} -2(3) + (-1) \\ 3(3) - 2(-1) \end{pmatrix} = \begin{pmatrix} -6 - 1 \\ 9 + 2 \end{pmatrix} = \begin{pmatrix} -7 \\ 11 \end{pmatrix}.$$

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Matrices from Linear Transformations

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Matrices from Linear Transformations

Generalizing this for any vector $\vec{v} = \begin{pmatrix} x \\ y \end{pmatrix} \in \mathbb{R}^2$ yields

$$T \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} ax + by \\ cx + dy \end{pmatrix},$$

where $a, b, c, d \in \mathbb{R}$.

This form of writing a linear transformation applied to a vector has a nice structure: the first component of the resulting vector is the dot product

$$\begin{pmatrix} a \\ b \end{pmatrix} \cdot \begin{pmatrix} x \\ y \end{pmatrix},$$

while the second component is the dot product

$$\begin{pmatrix} c \\ d \end{pmatrix} \cdot \begin{pmatrix} x \\ y \end{pmatrix}.$$

Matrices from Linear Transformations

We can collect the coefficients a, b, c and d together to a compact structure called a **matrix**:

$$M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

Then, we can define the product of that matrix with a vector $\vec{v} = \begin{pmatrix} x \\ y \end{pmatrix}$ as

$$M \cdot \vec{v} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} ax + by \\ cx + dy \end{pmatrix}.$$

Matrices from Linear Transformations

Note

In the matrix

$$M = \begin{pmatrix} a & b \\ c & d \end{pmatrix},$$

The column $\begin{pmatrix} a \\ c \end{pmatrix}$ shows us how $\hat{e}_1$ is transformed by the matrix M, while the column $\begin{pmatrix} b \\ d \end{pmatrix}$ shows us how $\hat{e}_2$ is transformed by the matrix.

Matrices from Linear Transformations

Of course, this can be generalized to any transformation $T : \mathbb{R}^n \to \mathbb{R}^n$ as

$$M = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix}.$$

The numbers a_{ij} are called the **elements** of the Matrix, where i is the **row** of the element, and j is the **column** of the element.

Matrices from Linear Transformations

Of course, this can be generalized to any transformation $T : \mathbb{R}^n \to \mathbb{R}^n$ as

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The numbers a_{ij} are called the **elements** of the Matrix, where i is the **row** of the element, and j is the **column** of the element.

In addition, each column of the matrix tells us how the respective standard basis vector is transformed.

Matrices from Linear Transformations

Example

The transformation $T : \mathbb{R}^3 \to \mathbb{R}^3$ which scales space by 2 in the x-direction, by 0.75 in the y-direction and by 1.5 in the z-direction, transforms the standard basis vectors $\hat{x}, \hat{y}$ and $\hat{z}$ as following:

Matrices from Linear Transformations

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$$T\left(\hat{x}\right) = \begin{pmatrix} 2 \\ 0 \\ 0 \end{pmatrix}, \; T\left(\hat{y}\right) = \begin{pmatrix} 0 \\ 0.75 \\ 0 \end{pmatrix}, \; T\left(\hat{z}\right) = \begin{pmatrix} 0 \\ 0 \\ 1.5 \end{pmatrix}.$$

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Thus, the corresponding matrix M is

$$M = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 0.75 & 0 \\ 0 & 0 & 1.5 \end{pmatrix}.$$

Matrices from Linear Transformations

Generalizing even further, a matrix representing a linear transformation of a type $T : \mathbb{R}^n \to \mathbb{R}^m$ is constructed from n column vectors, each of m components:

Matrices from Linear Transformations

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Example

The matrix

$$M = \begin{pmatrix} 1 & -3 & 7 \\ 2 & 0 & -5 \end{pmatrix}$$

takes vectors in $\mathbb{R}^3$ and transforms them to vectors in $\mathbb{R}^2$.

For example,

$$M \cdot \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} = \begin{pmatrix} 1 \cdot 1 - 3 \cdot 2 + 7 \cdot 3 \\ 2 \cdot 1 + 0 \cdot 2 - 5 \cdot 3 \end{pmatrix}$$

$$= \begin{pmatrix} 1 - 6 + 21 \\ 2 - 15 \end{pmatrix} = \begin{pmatrix} 16 \\ -13 \end{pmatrix}.$$

Types of Matrices

A matrix which represents a linear transformation $T : \mathbb{R}^n \to \mathbb{R}^n$ is called a **square matrix** (since it has $n \times n$ elements, i.e. n rows and n columns).

In a square matrix, the elements $a_{11}, a_{22}, \ldots, a_{nn}$ are called together the **principal diagonal** (also **main diagonal**) of the matrix.

$$\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{nn} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix}$$

A **diagonal matrix** is a matrix in which any element outside the main diagonal is zero.

Example

Some diagonal matrices:

$$\begin{pmatrix} 1 & 0 \\ 0 & -3 \end{pmatrix}, \quad \begin{pmatrix} -4 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 5 \end{pmatrix}, \quad \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 7 \end{pmatrix}.$$

Types of Matrices

An **upper triangular matrix** is a matrix in which all the elements **below** the main diagonal are equal to zero.

Similarly, a **lower triangular matrix** is a matrix in which all the elements **above** the main diagonal are equal to zero.

Types of Matrices

Example

An upper triangular matrix:

$$\begin{pmatrix} 1 & 4 & 3 & -1 \\ 0 & 2 & 3 & -1 \\ 0 & 0 & 7 & 7 \\ 0 & 0 & 0 & -2 \end{pmatrix}$$

A lower triangular matrix:

$$\begin{pmatrix} -1 & 0 & 0 & 0 \\ 2 & 5 & 0 & 0 \\ 6 & 4 & 7 & 0 \\ 1 & 0 & 3 & -2 \end{pmatrix}$$

Trace

The **trace** of a square matrix is the sum of its main diagonal elements, i.e. for an $n \times n$ matrix A with elements a_{ij},

$$\operatorname{tr}(A) = \sum_{i=1}^{n} a_{ii}.$$

Example

The trace of

$$A = \begin{pmatrix} 1 & 0 & 2 & -5 \\ 5 & 3 & 1 & 2 \\ -4 & 9 & 3 & -1 \\ 1 & 0 & 2 & 7 \end{pmatrix}$$

is $1 + 3 + 3 + 7 = 14$.

Transposing Matrices

An important operator that can be applied to matrices is the **transpose**. The transpose exchanges the rows of the matrix with its columns, i.e.

$$a_{ij} \xrightarrow{\text{transpose}} a_{ji}.$$

The notation for the transpose of a matrix A is $A^\top$.

Transposing Matrices

Example

$$\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix}^\top = \begin{pmatrix} 1 & 4 & 7 \\ 2 & 5 & 8 \\ 3 & 6 & 9 \end{pmatrix},$$

$$\begin{pmatrix} 0 & 1 & -1 \\ 2 & -3 & 5 \end{pmatrix}^\top = \begin{pmatrix} 0 & 2 \\ 1 & -3 \\ -1 & 5 \end{pmatrix}.$$

Note

In square matrices, the main diagonal elements stay at the same place when the matrix is transposed (since $a_{ii} = a_{ii}$). This also means that $\mathrm{tr}(A) = \mathrm{tr}\left(A^\top\right)$.

Transposing Matrices

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Note

The transpose of a transposed matrix is the original matrix, i.e.

$$\left(A^\top\right)^\top = A.$$

Adding Matrices

Like with vectors, addition of two matrices is done element-wise, i.e.

$$\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1m} \\ a_{21} & a_{22} & \cdots & a_{2m} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nm} \end{pmatrix} + \begin{pmatrix} b_{11} & b_{12} & \cdots & b_{1m} \\ b_{21} & b_{22} & \cdots & b_{2m} \\ \vdots & \vdots & \ddots & \vdots \\ b_{n1} & b_{n2} & \cdots & b_{nm} \end{pmatrix} =$$

$$\begin{pmatrix} a_{11}+b_{11} & a_{12}+b_{12} & \cdots & a_{1m}+b_{1m} \\ a_{21}+b_{21} & a_{22}+b_{22} & \cdots & a_{2m}+b_{2m} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1}+b_{n1} & a_{n2}+b_{n2} & \cdots & a_{nm}+b_{nm} \end{pmatrix}.$$

Adding Matrices

Example

$$\begin{pmatrix} 1 & 3 & -7 \\ 2 & 0 & 1 \\ 0 & -4 & 5 \end{pmatrix} + \begin{pmatrix} 0 & -2 & 1 \\ 3 & 2 & 3 \\ 5 & 6 & -1 \end{pmatrix} = \begin{pmatrix} 1 & 1 & -6 \\ 5 & 2 & 4 \\ 5 & 2 & 4 \end{pmatrix},$$

$$\begin{pmatrix} 2 & -1 & 0 \\ 1 & 5 & 9 \end{pmatrix} + \begin{pmatrix} 1 & 3 & 1 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 3 & 2 & 1 \\ 1 & 5 & 10 \end{pmatrix}.$$

Multiplying a Matrix by a Scalar

Also like with vectors, multiplying a matrix A by a scalar α results in multiplying each element a_{ij} of the matrix by α, i.e.

$$\alpha \cdot \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1m} \\ a_{21} & a_{22} & \cdots & a_{2m} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nm} \end{pmatrix} = \begin{pmatrix} \alpha \cdot a_{11} & \alpha \cdot a_{12} & \cdots & \alpha \cdot a_{1m} \\ \alpha \cdot a_{21} & \alpha \cdot a_{22} & \cdots & \alpha \cdot a_{2m} \\ \vdots & \vdots & \ddots & \vdots \\ \alpha \cdot a_{n1} & \alpha \cdot a_{n2} & \cdots & \alpha \cdot a_{nm} \end{pmatrix}.$$

Adding Matrices

Example

$$5 \cdot \begin{pmatrix} 1 & 3 & -7 \\ 2 & 0 & 1 \\ 0 & -4 & 5 \end{pmatrix} = \begin{pmatrix} 5 & 15 & -35 \\ 10 & 0 & 5 \\ 0 & -20 & 25 \end{pmatrix},$$

$$-\frac{1}{2} \cdot \begin{pmatrix} 2 & -6 & 2 \\ 0 & 0 & -2 \end{pmatrix} = \begin{pmatrix} -1 & 3 & -1 \\ 0 & 0 & 1 \end{pmatrix}.$$

The Identity Matrix

Since the columns in a matrix represent the transformation of the standard basis vectors, a matrix which is composed from the standard basis vectors in their original order does not change the space at all. We call such a matrix the **identity matrix**, denoted by I_n, where n is the dimentionality of the space.

$$I_n = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{pmatrix}.$$

The Identity Matrix

Example

$$I_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \; I_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \; I_4 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \ldots$$

The Identity Matrix

A shorthand way of writing the elements of the identity matrix is by using the **Kronecker delta**, which is defined as

$$\delta_{ij} = \begin{cases} 1 & \text{if } i = j, \\ 0 & \text{if } i \neq j. \end{cases}$$

Basic Matrices

Let's now go over the basic linear transformations $T : \mathbb{R}^2 \to \mathbb{R}^2$ mentioned in the previous chapter and construct a matrix for each one.

We will construct each matrix by looking at what effects does the respective transformation have on the basis vectors $\hat{x} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $\hat{y} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$, and then joining them together to form the matrix.

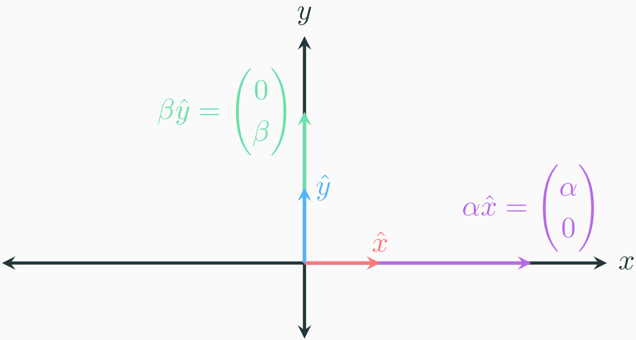
When possible, we will generalize to $\mathbb{R}^n$.

Scaling Matrices

Scaling in the x- and y-axes

Scaling in the x-axis by α should transform $\hat{x}$ to $\begin{pmatrix} \alpha \\ 0 \end{pmatrix}$.

Similarly, scaling in the y-axis by β should transform $\hat{y}$ to $\begin{pmatrix} 0 \\ \beta \end{pmatrix}$.



Scaling Matrices

Thus, a general scaling matrix in $\mathbb{R}^2$ is

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Thus, a general scaling matrix in $\mathbb{R}^2$ is

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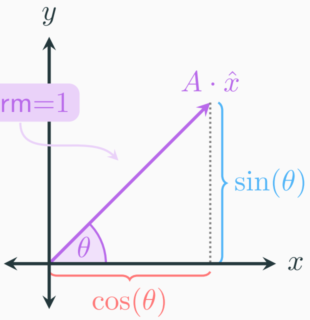
This can be generalized to $\mathbb{R}^n$ as a diagonal matrix

$$S = \begin{pmatrix} s_1 & 0 & \cdots & 0 \\ 0 & s_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & s_n \end{pmatrix}$$

Rotation Matrices

When rotating $\hat{x}$ by an angle θ, we get a vector that has norm 1 (because rotation doesn't change the norm) and thus the components

$$x = \cos(\theta),$$
$$y = \sin(\theta).$$



Rotation Matrices

Since $\hat{y}$ is 90° "ahead" of $\hat{x}$ (i.e. its angle to the x-axis is always 90° more than that of $\hat{x}$), we excpet the rotated $\hat{y}$ to have the components

$$x = \cos\left(\theta + 90°\right),$$
$$y = \sin\left(\theta + 90°\right).$$

Two trigonometric identities come in handy:

1. $\cos\left(\theta + 90°\right) = -\sin(\theta)$, and
2. $\sin\left(\theta + 90°\right) = \cos(\theta)$.

Thus, $\hat{y}$ transforms to $\begin{pmatrix} -\sin(\theta) \\ \cos(\theta) \end{pmatrix}$.

Rotation Matrices

Alltogether, in $\mathbb{R}^2$ the matrix representing a counter clock-wise rotation by θ around the origin is

$$\mathrm{Rot}(\theta) = \begin{pmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{pmatrix}.$$

Example

Rotation by 45° around the origin is given by

$$\mathrm{Rot}\,(45°) = \begin{pmatrix} \cos(45°) & -\sin(45°) \\ \sin(45°) & \cos(45°) \end{pmatrix} = \begin{pmatrix} \frac{2}{\sqrt{2}} & -\frac{2}{\sqrt{2}} \\ \frac{2}{\sqrt{2}} & \frac{2}{\sqrt{2}} \end{pmatrix}.$$

Applying this to the vector $\vec{v} = \begin{pmatrix} 1 \\ 3 \end{pmatrix}$ results in

$$\vec{u} = \mathrm{Rot}\,(45°) \cdot \vec{v} = \begin{pmatrix} \frac{2}{\sqrt{2}} & -\frac{2}{\sqrt{2}} \\ \frac{2}{\sqrt{2}} & \frac{2}{\sqrt{2}} \end{pmatrix} \begin{pmatrix} 1 \\ 3 \end{pmatrix} = \frac{\sqrt{2}}{2} \begin{pmatrix} 1-3 \\ 1+3 \end{pmatrix}$$

$$= \frac{\sqrt{2}}{2} \begin{pmatrix} -2 \\ 4 \end{pmatrix} = \begin{pmatrix} -\sqrt{2} \\ 2\sqrt{2} \end{pmatrix}.$$

Example

Let's verify our result.

The norm of the original vector is

$$\|\vec{v}\| = \sqrt{1^2 + 3^2} = \sqrt{10}.$$

The norm of the resulting vector is

$$\|\vec{u}\| = \sqrt{\left(-\sqrt{2}\right)^2 + \left(2\sqrt{2}\right)^2} = \sqrt{2+8} = \sqrt{10} = \|\vec{v}\|.$$

The cosine of the angle θ between the two vectors is

$$\cos(\theta) = \frac{\vec{v} \cdot \vec{u}}{\|\vec{v}\|\|\vec{u}\|} = \frac{-1\sqrt{2} + 3\left(2\sqrt{2}\right)}{\sqrt{10}\sqrt{10}} = \frac{5\sqrt{2}}{10} = \frac{\sqrt{2}}{2}.$$

Example

Let's verify our result.

The norm of the original vector is

$$\|\vec{v}\| = \sqrt{1^2 + 3^2} = \sqrt{10}.$$

The norm of the resulting vector is

$$\|\vec{u}\| = \sqrt{\left(-\sqrt{2}\right)^2 + \left(2\sqrt{2}\right)^2} = \sqrt{2+8} = \sqrt{10} = \|\vec{v}\|\,.$$

The cosine of the angle θ betwee $\cos\left(45^\circ\right) = \frac{\sqrt{2}}{2}$

$$\cos(\theta) = \frac{\vec{v}\cdot\vec{u}}{\|\vec{v}\|\|\vec{u}\|} = \frac{-1\sqrt{2} + 3\left(2\sqrt{2}\right)}{\sqrt{10}\sqrt{10}} = \frac{5\sqrt{2}}{10} = \frac{\sqrt{2}}{2}.$$

Rotation Matrices

In $\mathbb{R}^3$ the three matrices representing rotations by the angles θ, φ and ψ around the x, y- and z-axes, respectively, are

$$\mathrm{Rot}_x(\theta) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos(\theta) & -\sin(\theta) \\ 0 & \sin(\theta) & \cos(\theta) \end{pmatrix},$$

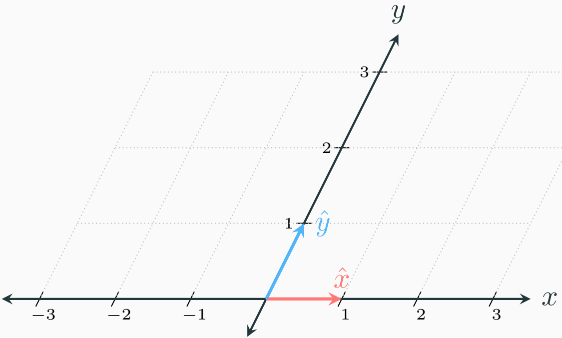
$$\mathrm{Rot}_y(\varphi) = \begin{pmatrix} \cos(\varphi) & 0 & \sin(\varphi) \\ 0 & 1 & 0 \\ -\sin(\varphi) & 0 & \cos(\varphi) \end{pmatrix},$$

$$\mathrm{Rot}_z(\psi) = \begin{pmatrix} \cos(\psi) & -\sin(\psi) & 0 \\ \sin(\psi) & \cos(\psi) & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Skew (Shear) Matrices

A shear transformation in the x-direction doesn't change $\hat{x}$, but adds some number k_x to the x-component of $\hat{y}$, i.e.

$$T(\hat{y}) = \begin{pmatrix} k_x \\ 1 \end{pmatrix}.$$



Skew (Shear) Matrices

Therefore, the matrix representing an x-direction shear by k_x is

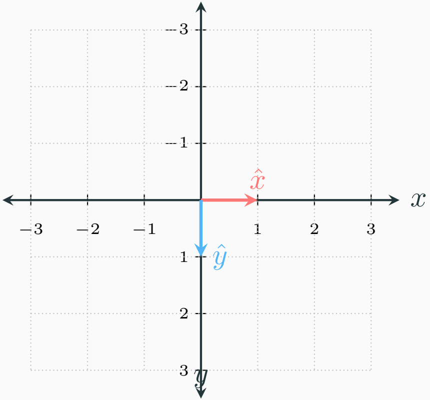
$$K = \begin{pmatrix} 1 & k_x \\ 0 & 1 \end{pmatrix}.$$

Similarily, a matrix representing a k_y shear in the y-direction is

$$K = \begin{pmatrix} 1 & 0 \\ k_y & 1 \end{pmatrix}.$$

Reflection Matrices

Reflection across the x-axis keeps $\hat{x}$ the same, and flips the y-component of $\hat{y}$.



Reflection Matrices

Thus, the matrix that reflects space across the x-axis is

$$\mathrm{Ref}_x = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Similarily, the matrix that reflects space across the y-axis is

$$\mathrm{Ref}_y = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Reflection Matrices

A general matrix which reflects space across a line going through the origin with slope m is

$$\mathrm{Ref}(\theta) = \begin{pmatrix} \cos(2\theta) & \sin(2\theta) \\ \sin(2\theta) & -\cos(2\theta) \end{pmatrix},$$

where $\theta = \arctan(m)$.

Note

The derivition of $\mathrm{Ref}(\theta)$ from Ref_x requires a concept which we will introduce later in this chapter, and thus will not be given here.

Determinants

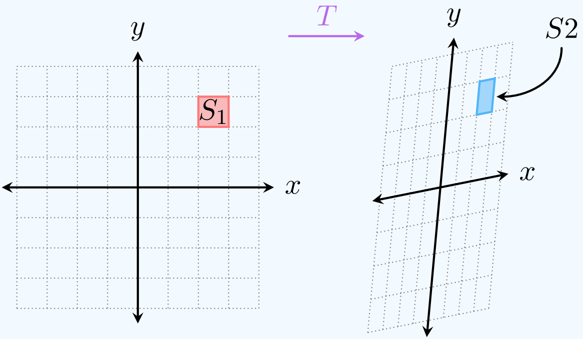
The **determinant** of an $n \times n$ matrix A, denoted $|A|$ or $\det(A)$, is a measurement of how volumes scale when applying the transformation represented by A to a space $\mathbb{R}^n$.

Note

A determinant of a matrix can be negative, while volumes (technically speaking) must be non-negative. We will see the meaning of a negative determinant later in this chapter.

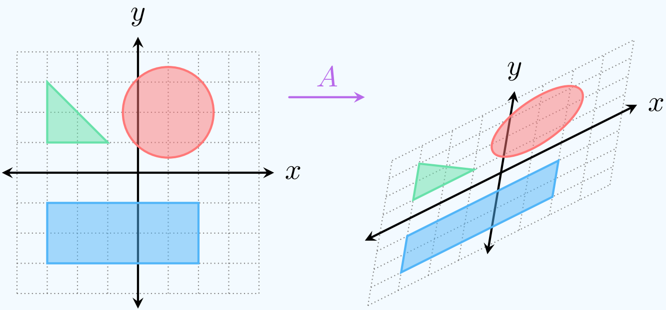
Example

The determinant of a 2×2 matrix A measures the change in **area** after applying A to $\mathbb{R}^2$. In the following example, the area denoted by S_1 is transformed into the area S_2 after application of the transformation (i.e. $|A| = \frac{S_2}{S_1}$).



Example

Since linear transformations scale all areas (volumes) by the same amount, in the following example, the determinant is equal to the ratio between the areas of each shape after and before application of the transformation (i.e. $|A| = \frac{S_{\text{after}}}{S_{\text{before}}}$).

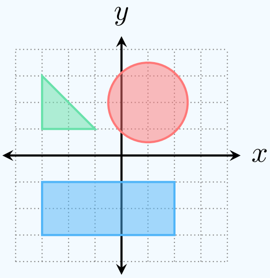


Determinants

When the image of a linear transformation $T : \mathbb{R}^n \to \mathbb{R}^n$ can be spanned by $m < n$ vectors (i.e. when it "loses" a dimension or more) the determinant of the matrix representing it is zero.

Example

A sequence of linear transformations with the determinants of their respective matrices going to zero. When $|A| = 0$ all of space is "squished" into a line, i.e. it has zero area.

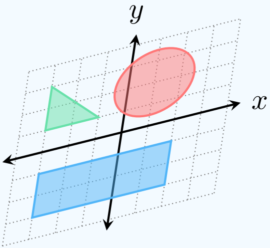


$|A| = 1$

Determinants

Example

A sequence of linear transformations with the determinants of their respective matrices going to zero. When $|A| = 0$ all of space is "squished" into a line, i.e. it has zero area.

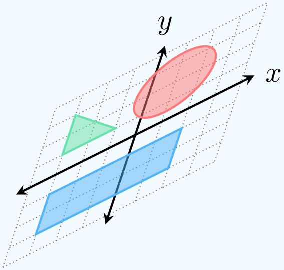


$$|A| = 0.8$$

Determinants

Example

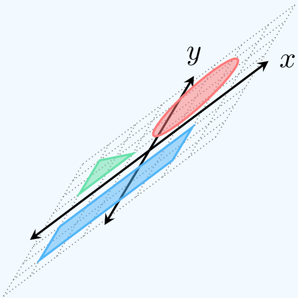
A sequence of linear transformations with the determinants of their respective matrices going to zero. When $|A| = 0$ all of space is "squished" into a line, i.e. it has zero area.



$$|A| = 0.625$$

Example

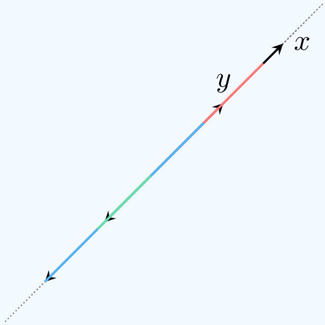
A sequence of linear transformations with the determinants of their respective matrices going to zero. When $|A| = 0$ all of space is "squished" into a line, i.e. it has zero area.



$|A| = 0.35$

Example

A sequence of linear transformations with the determinants of their respective matrices going to zero. When $|A| = 0$ all of space is "squished" into a line, i.e. it has zero area.



$$|A| = 0$$

Determinants

Square matrices of size 3×3 (representing transformations of the type $T : \mathbb{R}^3 \to \mathbb{R}^3$) have zero determinants when they "squish" 3D-space into a single plane, a line or a point.

A matrix with zero determinant has columns that are **linearly dependent**. This is because the columns of a matrix represent the transformations of the basis vectors $\hat{e}_1, \hat{e}_2, \cdots, \hat{e}_n$. If this set is linearly dependent, the space the vectors span has a lower dimentionality than the original space.

Example

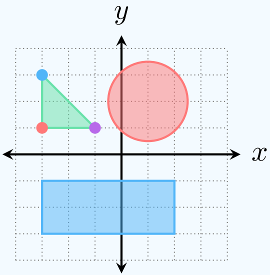
The following matrix has a zero determinant:

$$\begin{pmatrix} 1 & 0 & 2 \\ 3 & 1 & 5 \\ 0 & 4 & -4 \end{pmatrix} \Rightarrow 2\begin{pmatrix} 1 \\ 3 \\ 0 \end{pmatrix} - \begin{pmatrix} 0 \\ 1 \\ 4 \end{pmatrix} = \begin{pmatrix} 2 \\ 5 \\ -4 \end{pmatrix}.$$

Determinants

Matrices can have negative determinants, which means that after application of their respective transformation, the space chages its **orientation**.

Example

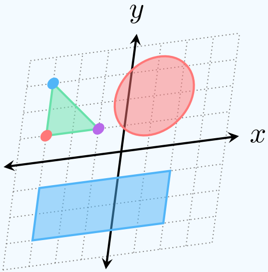


$|A| = 1$

Determinants

Matrices can have negative determinants, which means that after application of their respective transformation, the space chages its **orientation**.

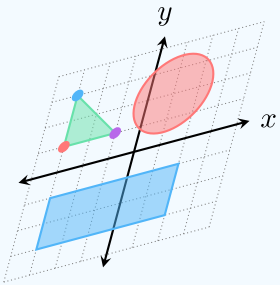
Example



$$|A| = 0.97$$

Matrices can have negative determinants, which means that after application of their respective transformation, the space chages its **orientation**.

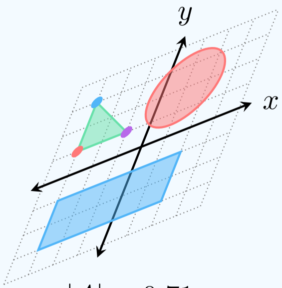
Example



$$|A| = 0.87$$

Matrices can have negative determinants, which means that after application of their respective transformation, the space chages its **orientation**.

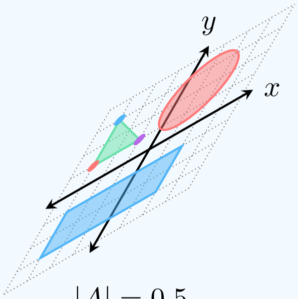
Example



$|A| = 0.71$

Matrices can have negative determinants, which means that after application of their respective transformation, the space chages its **orientation** .

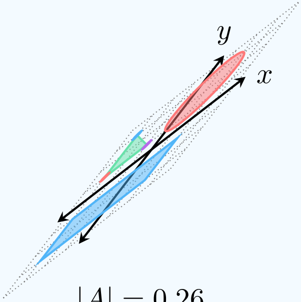
Example



$|A| = 0.5$

Matrices can have negative determinants, which means that after application of their respective transformation, the space chages its **orientation**.

Example

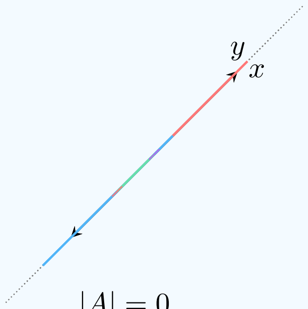


$|A| = 0.26$

Determinants

Matrices can have negative determinants, which means that after application of their respective transformation, the space chages its **orientation** .

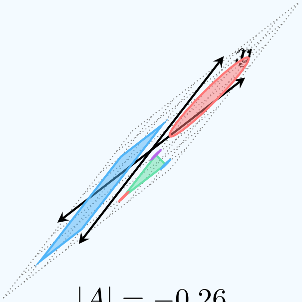
Example



$|A| = 0$

Matrices can have negative determinants, which means that after application of their respective transformation, the space chages its **orientation**.

Example

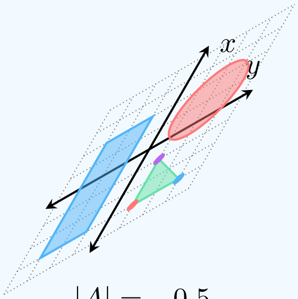


$$|A| = -0.26$$

Determinants

Matrices can have negative determinants, which means that after application of their respective transformation, the space chages its **orientation**.

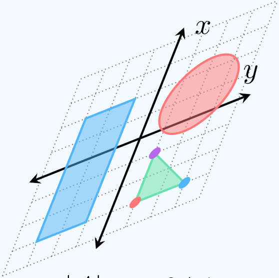
Example



$$|A| = -0.5$$

Matrices can have negative determinants, which means that after application of their respective transformation, the space chages its **orientation**.

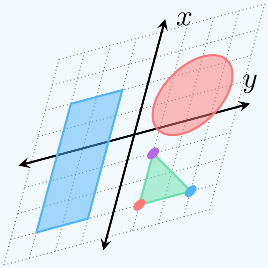
Example



$$|A| = -0.71$$

Matrices can have negative determinants, which means that after application of their respective transformation, the space chages its **orientation**.

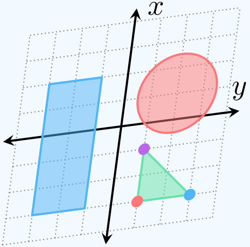
Example



$$|A| = -0.87$$

Matrices can have negative determinants, which means that after application of their respective transformation, the space chages its **orientation**.

Example

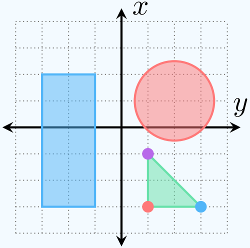


$$|A| = -0.97$$

Determinants

Matrices can have negative determinants, which means that after application of their respective transformation, the space chages its **orientation**.

Example

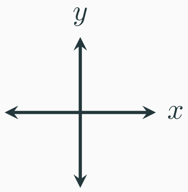


$$|A| = -1$$

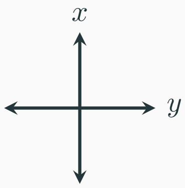
Determinants

The orientation of $\mathbb{R}^2$ is determined by the relative direction between the x- and y-axes:

- If the x-axis is to the **right** of the y-axis, the space is **right-handed**.
- If the x-axis is to the **left** of the y-axis, the space is **left-handed**.



Right-handed

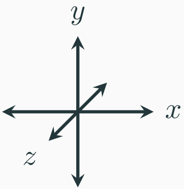


Left-handed

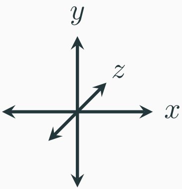
Determinants

The orientation of $\mathbb{R}^3$ is determined by the right-hand rule, similar to the cross-product:

- If $\hat{x} \times \hat{y} = \hat{z}$, the space is **right-handed**.
- If $\hat{x} \times \hat{z} = \hat{y}$, the space is **left-handed**.



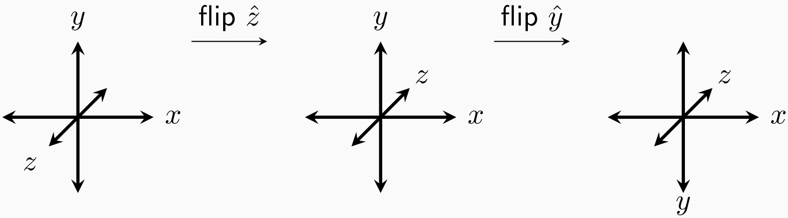
Right-handed



Left-handed

Any flip of an **odd** number of axes **flips** the orientation of a space.

Any flip of an **even** number of axes **keeps** the orientation of a space.



Right-handed Left-handed Right-handed

Determinants

Determinants can be calculated numerically directly from matrices.

The determinant of a 2×2 matrix

$$A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$$

is

$$|A| = a_{11}a_{22} - a_{12}a_{21}.$$

Determinants

Example

Some 2×2 matrices and their determinants:

$$\begin{vmatrix} 1 & 2 \\ -3 & 1 \end{vmatrix} = 1 \cdot 1 - 2 \cdot (-3) = 1 + 6 = 7.$$

$$\begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} = 1 \cdot 1 - 0 \cdot 0 = 1.$$

$$\begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix} = 0 \cdot 0 - 1 \cdot 1 = -1.$$

$$\begin{vmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{vmatrix} = \cos(\theta)^2 + \sin(\theta)^2 = 1.$$

For calculating a the determinant of a 3×3 matrix we introduce a new concept, a **minor of a matrix**:

Definition

The i, j-minor of a matrix A, denoted m_{ij}, is the determinant of a matrix produced when the i-th row and j-th column of A are removed.

Determinants

Example

Two example minors of the matrix $A = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix}$:

$$m_{11} = \begin{vmatrix} \text{[redacted]} & & \\ & 5 & 6 \\ & 8 & 9 \end{vmatrix} = 5 \cdot 9 - 6 \cdot 8 = 45 - 48 = -3.$$

$$m_{32} = \begin{vmatrix} 1 & \text{[redacted]} & 3 \\ 4 & & 6 \\ & & \end{vmatrix} = 1 \cdot 6 - 3 \cdot 4 = 6 - 12 = -6.$$

Determinants

The determinant of a 3×3 matrix

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$$

is

$$|A| = a_{11}m_{11} - a_{12}m_{12} + a_{13}m_{13},$$

where m_{ij} is the i, j-minor of A.

Example

Let's calculate the determinant of the matrix

$$A = \begin{pmatrix} -1 & 2 & 0 \\ 3 & 1 & 5 \\ 2 & 6 & -9 \end{pmatrix}.$$

1. $m_{11} = \begin{vmatrix} 1 & 5 \\ 6 & -9 \end{vmatrix} = 1 \cdot (-9) - 5 \cdot 6 = -9 - 30 = -39.$
2. $m_{12} = \begin{vmatrix} 3 & 5 \\ 2 & -9 \end{vmatrix} = 3 \cdot (-9) - 5 \cdot 2 = -27 - 10 = -37.$
3. $m_{13} = \begin{vmatrix} 3 & 1 \\ 2 & 6 \end{vmatrix} = 3 \cdot 6 - 1 \cdot 2 = 18 - 2 = 16.$

Example

Thus,

$$\begin{aligned}|A| &= a_{11}m_{11} - a_{12}m_{12} + a_{13}m_{13}\\ &= -1\cdot(-39) - 2\cdot(-37) + 0\cdot 16\\ &= 39 + 74\\ &= 113.\end{aligned}$$

Determinants

The determinant of higher order $n \times n$ matrices proceeds recursively from that of $(n-1) \times (n-1)$ matrices.

The definition will not be given here[1].

[1] and such determinants will not be in the exam in any way, don't worry.

Matrix-Vector Product

As we saw in the beginning of the chapter, multiplying a matrix with a vector is essencially a way to apply the transformation represented by the matrix on the vector. This was developed for vectors in $\mathbb{R}^2$ and 2×2 matrices.

We will now review a general product of an $n \times m$ matrix by an m-dimentional vector.

Matrix-Vector Product

The product of an $n \times m$ matrix A and an m-dimentional vector $\vec{u}$ is an n-dimentional vector $\vec{v}$, with the i-th component of $\vec{v}$ being

$$v_i = A_i \cdot \vec{u},$$

where A_i is the i-th row of A.

$$\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1m} \\ a_{21} & a_{22} & \cdots & a_{2m} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nm} \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \\ \vdots \\ u_m \end{pmatrix} = \begin{pmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{pmatrix}$$

Matrix-Vector Product

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$$v_i = A_i \cdot \vec{u},$$

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$$\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1m} \\ a_{21} & a_{22} & \cdots & a_{2m} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nm} \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \\ \vdots \\ u_m \end{pmatrix} = \begin{pmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{pmatrix}$$

Matrix-Vector Product

Example

$$\begin{pmatrix} 1 & -3 & 0 \\ 4 & 2 & 7 \\ -2 & 3 & -5 \\ 6 & 0 & -9 \end{pmatrix} \begin{pmatrix} 7 \\ -1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \cdot 7 + (-3) \cdot (-1) + 0 \cdot 0 \\ 4 \cdot 7 + 2 \cdot (-1) + 7 \cdot 0 \\ -2 \cdot 7 + 3 \cdot (-1) + (-5) \cdot 0 \\ 6 \cdot 7 + 0 \cdot (-1) + (-9) \cdot 0 \end{pmatrix}$$

$$= \begin{pmatrix} 10 \\ 26 \\ -17 \\ 42 \end{pmatrix}.$$

Matrix-Matrix Product

Matrices can be multiplied together. The result of the product of two matrices A and B, $A \cdot B$, is a matrix C, in which the i-th column is the product of the matrix A by the i-th column of the matrix B.

Example

$$\begin{pmatrix} 3 & -2 & 8 \\ -2 & -4 & 0 \\ -9 & -2 & 3 \end{pmatrix} \cdot \begin{pmatrix} 7 & -8 & -2 \\ -8 & 1 & -6 \\ 1 & 7 & -3 \end{pmatrix} = \begin{pmatrix} 45 & 30 & -18 \\ 18 & 12 & 28 \\ -44 & 91 & 21 \end{pmatrix}.$$

Matrix-Matrix Product

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Matrix-Matrix Product

Another way to formulize the matrix-matrix product of two matrices A and B is by considering the dot product of rows of A with columns of B, i.e.

Definition

The product $C = AB$ of two matrices A and B is a matrix in which the element c_{ij} is

$$c_{ij} = A^i \cdot B_j,$$

where A^i is the i-th row of A (considered as a vector), and B_j is the j-th column of B (considered as a vector).

Matrix-Matrix Product

Of course, for the above matrix-matrix product to be defined, the number of elements in each column vector of B must be equal to the number of columns in A, i.e.

Definition

For the matrix-matrix product $C = AB$ to be defined, the dimensions of A and B must respectively be

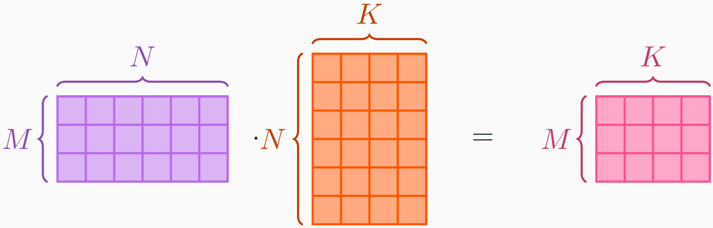
$$m \times n,\ n \times k.$$

The resulting matrix C has dimensions

$$m \times k.$$

Matrix-Matrix Product

Graphically:



$$A \cdot B = C$$

Matrix-Matrix Product

Matrix-matrix product (even for square matrices of the same dimensions) is **non-commutative**, i.e. for most matrices A, B

$$AB \neq BA.$$

Example

For $A = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}$ and $B = \begin{pmatrix} 0 & 3 \\ -1 & 1 \end{pmatrix}$:

$$AB = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 0 & 3 \\ -1 & 1 \end{pmatrix} = \begin{pmatrix} -2 & 5 \\ -1 & 1 \end{pmatrix},$$

$$BA = \begin{pmatrix} 0 & 3 \\ -1 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 3 \\ -1 & -1 \end{pmatrix} \neq AB.$$

Matrix-Matrix Product

For two matrices A and B, where A represents the linear transformation T_A and B represents the linear transformation T_B, the product $C = AB$ represents the composit transformation $T_C = T_A \circ T_B$.

Matrix-Matrix Product

Example

The matrix $A = \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}$ represents a scaling of space by 2 in the x-direction. The matrix $B = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ represents a rotation of space by 90° counter clock-wise. The product

$$C = AB = \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & -2 \\ 1 & 0 \end{pmatrix}$$

represents a 90° counter clock-wise rotation, followed by scaling by 2 in the x-direction.

Matrix-Matrix Product

Note

The product $D = BA$ represents the composition of the transformations in the opposite order: first scaling space by 2 in the x-direction, followed by a 90° counter clock-wise rotation, i.e. $T_D = T_B \circ T_A$.

Matrix-Matrix Product

Since the product of two matrices is in fact composition of the respective transformations they represent, the determinant of the product of two matrices A and B is simply the product of the determinants of the two matrices, i.e. if $C = AB$ then

$$|C| = |A||B|.$$

Example

For $A = \begin{pmatrix} 1 & -3 \\ 0 & 4 \end{pmatrix}$ and $B = \begin{pmatrix} 0 & 1 \\ 2 & 2 \end{pmatrix}$,

$$\begin{aligned} |A| &= 1 \cdot 4 - \cancel{(-3) \cdot 0} = 4, \\ |B| &= \cancel{0 \cdot 2} - 1 \cdot 2 = -2, \\ |AB| &= \left| \begin{pmatrix} -6 & -5 \\ 8 & 8 \end{pmatrix} \right| = -6 \cdot 8 - (-5) \cdot 8 \\ &= -48 + 40 = -8 = |A| \cdot |B|. \end{aligned}$$

Matrix-Matrix Product

The transpose of the product of two matrices is the opposite product of the transpose of each matrix, i.e.

$$(AB)^\top = B^\top A^\top.$$

Example

For $A = \begin{pmatrix} 1 & -3 \\ 0 & 4 \end{pmatrix}$ and $B = \begin{pmatrix} 0 & 1 \\ 2 & 2 \end{pmatrix}$,

$$AB = \begin{pmatrix} -6 & -5 \\ 8 & 8 \end{pmatrix},$$

and so

$$(AB)^\top = \begin{pmatrix} -6 & 8 \\ -5 & 8 \end{pmatrix}.$$

Matrix-Matrix Product

Example

On the other hand,

$$
\begin{aligned}
B^\top A^\top &= \begin{pmatrix} 0 & 2 \\ 1 & 2 \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 \\ -3 & 4 \end{pmatrix} \\
&= \begin{pmatrix} -6 & 8 \\ -5 & 8 \end{pmatrix} \\
&= (AB)^\top .
\end{aligned}
$$

Matrix-Matrix Product

The trace of a Matrix-Matrix Product AB does not depend on the order of multiplication, i.e.

$$\operatorname{tr}(AB) = \operatorname{tr}(BA).$$

Challenge

Prove the above statement.

Inverse Matrices

An inverse matrix A^{-1} of a matrix A is a matrix for which

$$AA^{-1} = A^{-1}A = I.$$

Not every matrix A has an inverse matrix; if $|A| = 0$, then A "loses" at least one dimension, which results in more than one vector in its image being connected to by more than one vector in its domain. Therefore, A^{-1} does not exist.

More formally:

$$\exists A^{-1} \Leftrightarrow |A| \neq 0.$$

Inverse Matrices

Example

The matrix

$$A = \begin{pmatrix} 3 & -1 & 1 \\ 0 & -1 & -2 \\ 1 & 0 & 1 \end{pmatrix}$$

has zero determinant, since $A_3 = A_1 + 2A_2$. Therefore, A^{-1} doesn't exist.

Inverse Matrices

Example

The matrix

$$B = \begin{pmatrix} 2 & -4 & 2 \\ 0 & 1 & 0 \\ 1 & 5 & 3 \end{pmatrix}$$

has a determinant of 4, and thus B^{-1} exists, and is equal to

$$B^{-1} = \begin{pmatrix} 0.75 & 5.5 & -0.5 \\ 0 & 1 & 0 \\ -0.25 & -3.5 & 0.5 \end{pmatrix}.$$

Inverse Matrices

A matrix that **does not** have an inverse matrix is called a **singular matrix** (also a **degenerate matrix**). A matrix with an inverse is called an **invertible matrix** (also a **nonsingular matrix** and a **nondegenerate matrix**).

Inverse Matrices

A matrix that **does not** have an inverse matrix is called a **singular matrix** (also a **degenerate matrix**). A matrix with an inverse is called an **invertible matrix** (also a **nonsingular matrix** and a **nondegenerate matrix**).

Finding the inverse of a nonsingular matrix is not straight-forward, and there are many methods that were developed for this purpose. Some examples are **Gaussian elimination**, **Newton's method** and **Eigendecomposition**. We will not discuss these methods in these lectures, and will only show the practical inversion of 2×2 matrices.

Inverse Matrices

The inverse of a nonsingular 2×2 matrix

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

is

$$\begin{aligned} A^{-1} &= \frac{1}{|A|} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}. \\ &= \frac{1}{ad - bc} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}. \end{aligned}$$

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$$= \frac{1}{|ad - bc|} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}.$$

equal 0 when A is singular

Example

The inverse of the matrix

$$A = \begin{pmatrix} 3 & -1 \\ 0 & 2 \end{pmatrix}$$

is

$$\begin{aligned} A^{-1} &= \frac{1}{3 \cdot 2 - \cancel{(-1) \cdot 0}} \begin{pmatrix} 2 & 1 \\ 0 & 3 \end{pmatrix} \\ &= \frac{1}{6} \begin{pmatrix} 2 & 1 \\ 0 & 3 \end{pmatrix}. \end{aligned}$$

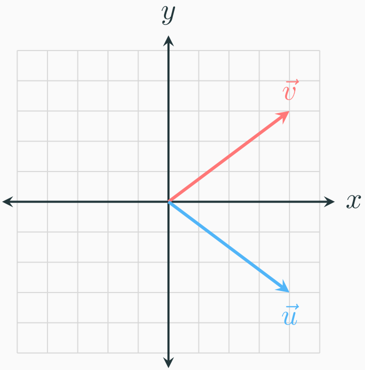
Inverse Matrices

Since multiplying a nonsingular matrix by its inverse results in the identity matrix, inverse matrices represent the inverse transformations, i.e. if A represents the transformation T, then A^{-1} represents the transformation T^{-1}.

Inverse Matrices

Let's use inverse matrices to construct the general 2×2 reflection matrix.

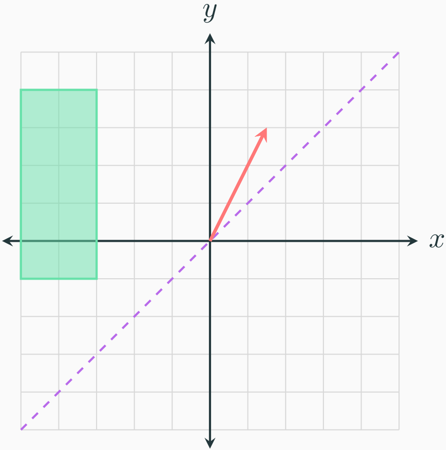
We know how to reflect across the x-axis: this simply means inverting the y-component.



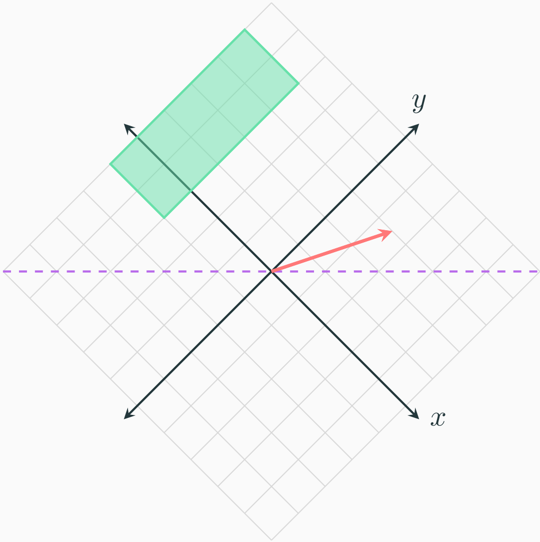
For a reflection across any line going through the origin, we can do the following:

1. Rotate space to align the reflection line with the horizontal direction.
2. Reflect across the horizontal direction (i.e. flip the y-components of all vectors).
3. Rotate space back, i.e. by the opposite angle as before.

Inverse Matrices



Inverse Matrices



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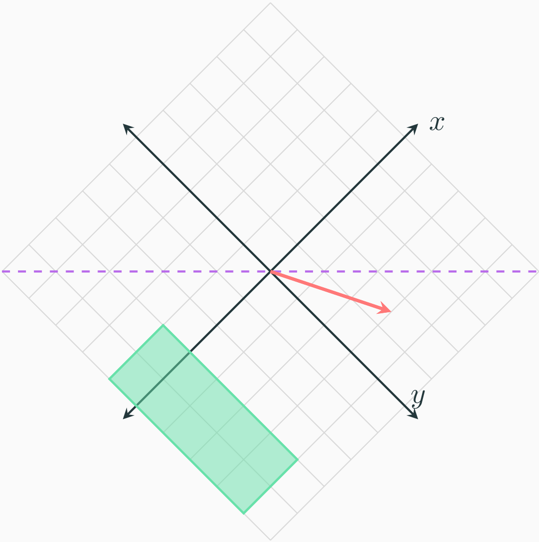
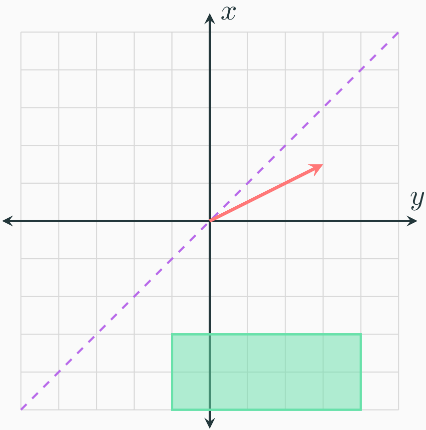
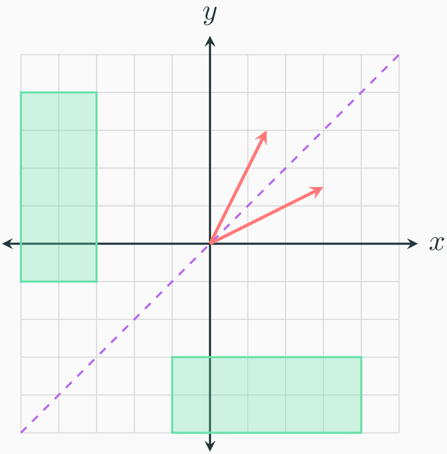


Inverse Matrices



Inverse Matrices



Inverse Matrices

Writing all these operations as a matrix product:

3. Rotate back — 2. Flip vertically — 1. Rotate

$$\mathrm{Ref}(\theta) = \begin{pmatrix} \cos(\theta) & \sin(\theta) \\ -\sin(\theta) & \cos(\theta) \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{pmatrix}$$

$$= \begin{pmatrix} \cos(2\theta) & \sin(2\theta) \\ \sin(2\theta) & -\cos(2\theta) \end{pmatrix}.$$

Kernel, Null Space

The **kernel** (also **null space**) of a linear transformation $T : \mathbb{R}^n \to \mathbb{R}^m$ is the set of all vectors that the linear transformation maps to the zero vector in $\mathbb{R}^m$, i.e.

$$\ker(T) = \left\{ \vec{v} \in \mathbb{R}^n \mid T\left(\vec{v}\right) = \vec{0} \right\}.$$

The kernel of a matrix A is the kernel of the linear transformation it represents.

Kernel, Null Space

Any linear combination of vectors in the kernel of a transformation T is also in its kernel.

Proof

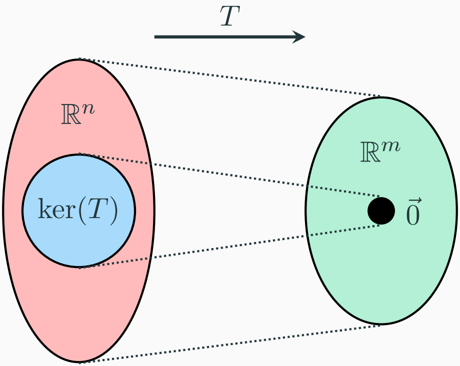
Suppose that $\vec{v} = \sum_{i=1}^{k} \alpha_k \vec{w}_k$, where $\vec{w}_i \in \ker(T)$ and $\alpha_i \in \mathbb{R}$.
Due to the linearity of T,

$$\begin{aligned} T(\vec{v}) &= T(\alpha_1 \vec{w}_1 + \alpha_2 \vec{w}_2 + \cdots + \alpha_k \vec{v}_k) \\ &= \alpha_1 T(\vec{w}_1) + \alpha_2 T(\vec{w}_2) + \cdots + \alpha_k T(\vec{w}_k) \\ &= \alpha_1 \vec{0} + \alpha_2 \vec{0} + \cdots + \alpha_k \vec{0} \\ &= \vec{0}. \end{aligned}$$

Therefore, $\vec{v} \in \ker(T)$.

Kernel, Null Space

This means that $\ker(T)$ is a subspace of the domain of T.



Kernel, Null Space

When refering to the matrix A which represents the transformation T, the kernel is refered to as the **null space** of A (denoted $\mathrm{Null}(A)$).

The dimension of $\mathrm{Null}(A)$ is called the **nullity** of A.

Kernel, Null Space

The **rank** of a matrix is the dimentionality of its **column space**, which is the space spanned by its columns when regarded as vectors.

If the dimentionality of the column space of an $n \times n$ matrix is smaller than n, the matrix is singular (i.e. it has a zero determinant).

Kernel, Null Space

For a matrix A of dimension $n \times n$,

$$\mathrm{rank}(A) = n \Leftrightarrow |A| \neq 0.$$