GEORG-AUGUST-UNIVERSITÄT
GÖTTINGEN

Mathematics and Computer Science (B.MES.108)
Summer Semester, 2020
Part 1: Linear Algebra for Non-Mathematicians

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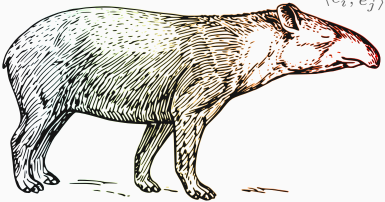
$(AB)^{\top} = B^{\top} A^{\top}$ $\quad \mathbb{R}^n \xrightarrow{T} \mathbb{R}^m$

$\vec{v} = \sum_{i=1}^{n} \alpha_i \hat{e}_i$ $\quad A = Q \Lambda Q^{-1}$

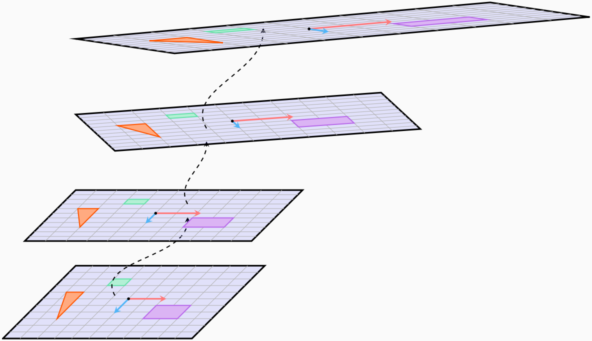
$\mathrm{Rot}(\theta) = \begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix}$ $\quad A\vec{v} = \lambda \vec{v}$

$T(\alpha \vec{u} + \beta \vec{v}) = \alpha T(\vec{u}) + \beta T(\vec{v})$

$\langle \hat{e}_i, \hat{e}_j \rangle = \delta_{ij}$



Chapter 3: Linear Transformations



Linear Transformations: Definition

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Therefore, f is linear.

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Therefore, g is **NOT** linear.

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Challenge

Check whether the function g from before complies with the 2nd criterion (additivity).

Example

Is the function $h(x) = x^2$ linear? Let's check additivity first:

$$h(x+y) = (x+y)^2 = x^2+2xy+y^2, \quad h(x)+h(y) = x^2+y^2.$$

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On the other hand,

$$\begin{aligned} h(x)+h(y) &= h(1)+h(-2) = 1^2+(-2)^2 \\ &= 1+4 = 5 \neq 1. \end{aligned}$$

Thus, h is also **NOT** linear.

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Check whether h fulfills the 1st criterion (scalability).

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We can combine both criteria to a single test for linearity of a transformation T:

Definition

A transformation $T : A \to B$ is linear, if for all $x, y \in A$ and $\alpha, \beta \in \mathbb{R}$

$$T\left(\alpha x + \beta y\right) = \alpha T(x) + \beta T(y).$$

Transforming Vectors

Vectors can also be transformed, specifically by functions of the type $T : \mathbb{R}^n \to \mathbb{R}^m$, with $n, m \in \mathbb{N}$.

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In this course we will mostly concentrate on transformations of the types

- $T : \mathbb{R}^2 \to \mathbb{R}^2$ and
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However, everything we learn about these transformations is applicable for any linear transformation, **regardless of its dimensionality**.

Transforming Vectors

Example

Applying the transformation $T : \mathbb{R}^2 \to \mathbb{R}^2$,

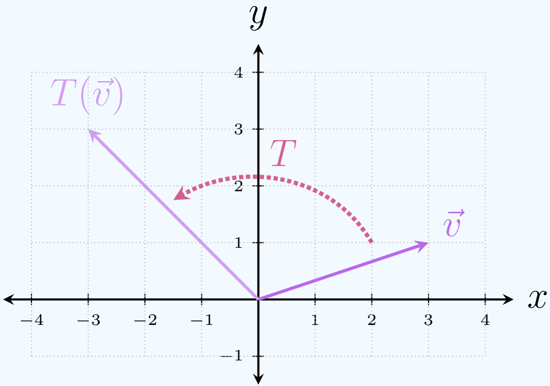
$$T \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -x \\ 3y \end{pmatrix}$$

to the vector $\vec{u} = \begin{pmatrix} 3 \\ 1 \end{pmatrix}$:

$$T \begin{pmatrix} 3 \\ 1 \end{pmatrix} = \begin{pmatrix} -3 \\ 3 \cdot 1 \end{pmatrix} = \begin{pmatrix} -3 \\ 3 \end{pmatrix}.$$

Example

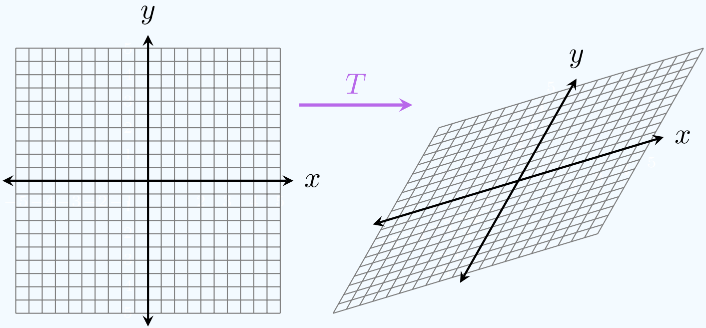
Graphically, the transformation looks as follows:



Transforming Spaces

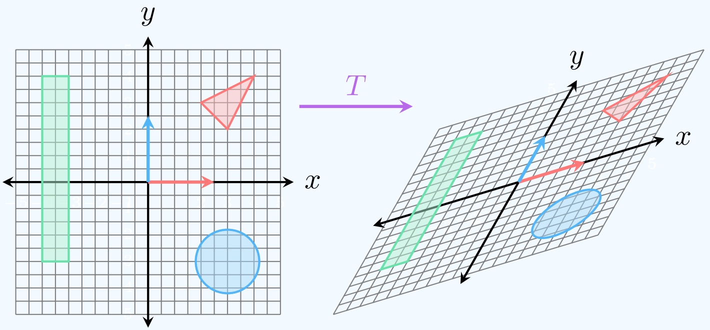
We can visualize the way an entire space is transformed by a transformation T by looking at how the axes and main gridlines of the space are transformed.

Example



This method also allows us to see how the basis vectors $\hat{x}$ and $\hat{y}$ are transformed by linear transformations, and also the transformations of shapes (all this will come in handy later).

Example



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Challenge

Show that these properties can be derived from the definition of linear transformations.

Types of Linear Transformations

Many linear transformations $T : \mathbb{R}^2 \to \mathbb{R}^2$ can be created by composition of two or more of the following basic transformations:

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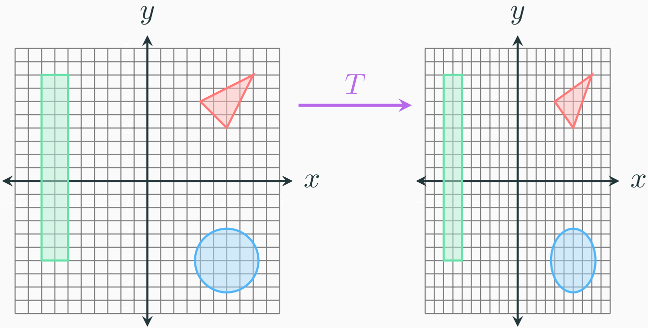
Scaling in the x-axis



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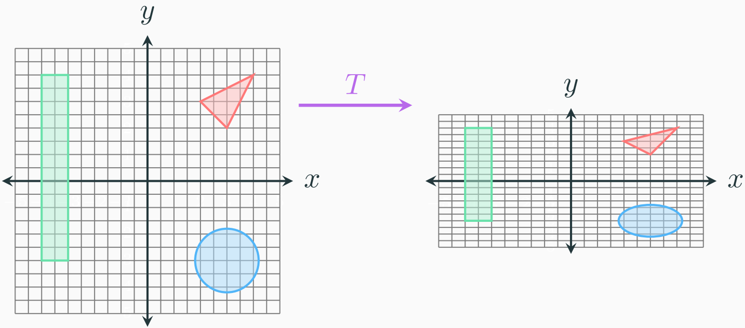

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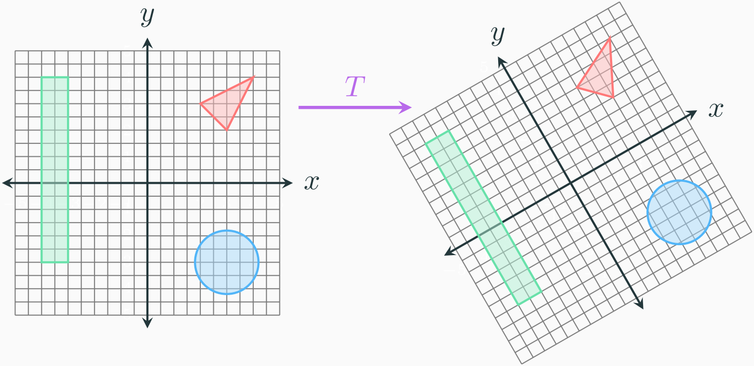
Rotation around the origin



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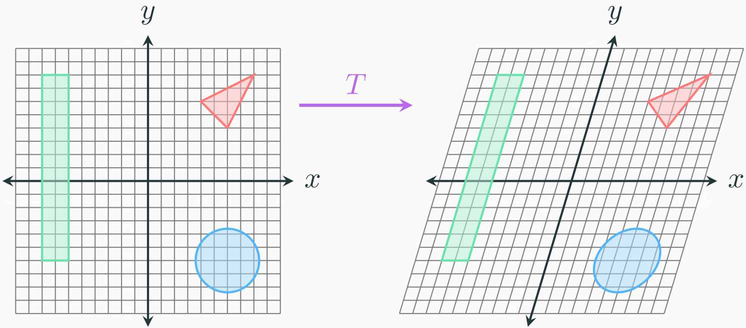

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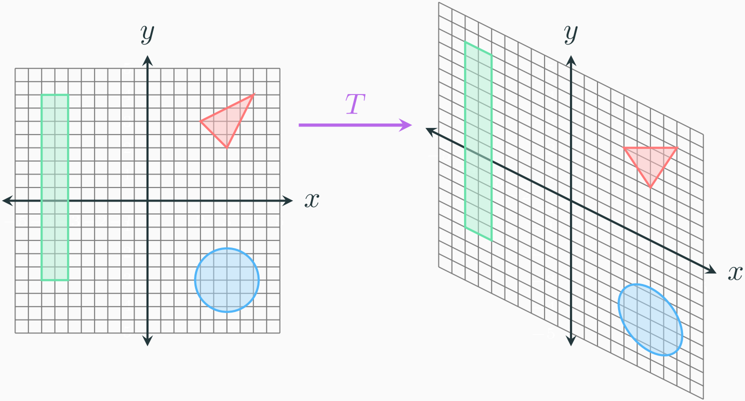

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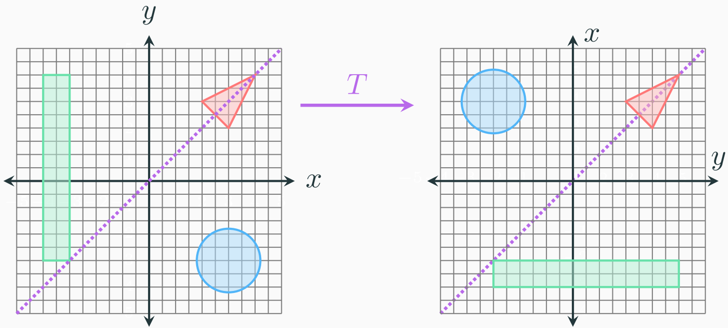
Reflection by a line going through the origin



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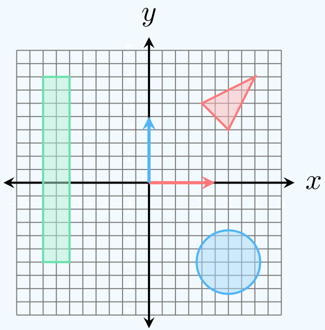

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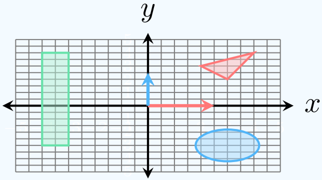

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