



# Mathematics and Computer Science (B.MES.108)

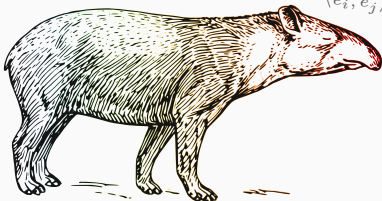
## Summer Semester, 2020

### Part 1: Linear Algebra for Non-Mathematicians

---

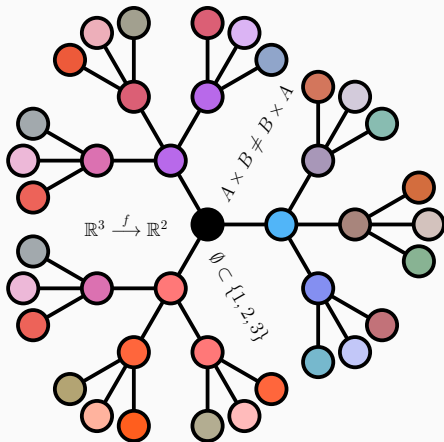
Peleg Bar Sapir

$$(AB)^T = B^T A^T$$
$$\vec{v} = \sum_{i=1}^n \alpha_i \hat{e}_i \quad \mathbb{R}^n \xrightarrow{T} \mathbb{R}^m$$
$$A = Q\Lambda Q^{-1}$$
$$\text{Rot}(\theta) = \begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix} \quad A\vec{v} = \lambda\vec{v}$$
$$T(\alpha\vec{u} + \beta\vec{v}) = \alpha T(\vec{u}) + \beta T(\vec{v})$$
$$\langle \hat{e}_i, \hat{e}_j \rangle = \delta_{ij}$$



# Chapter 1: Introduction

---



# Mathematical Propositions

## Definition

A **mathematical proposition** is a statement that can be either **true** or **false**.

# Mathematical Propositions

## Definition

A **mathematical proposition** is a statement that can be either **true** or **false**.

## Example

- The Moon's radius is smaller than the Earth's radius (**true**)

# Mathematical Propositions

## Definition

A **mathematical proposition** is a statement that can be either **true** or **false**.

## Example

- The Moon's radius is smaller than the Earth's radius (**true**)
- $1 + 2 = 3$  (**true**)

# Mathematical Propositions

## Definition

A **mathematical proposition** is a statement that can be either **true** or **false**.

## Example

- The Moon's radius is smaller than the Earth's radius (**true**)
- $1 + 2 = 3$  (**true**)
- Protons have no electric charge (**false**)

# Mathematical Propositions

## Definition

A **mathematical proposition** is a statement that can be either **true** or **false**.

## Example

- The Moon's radius is smaller than the Earth's radius (**true**)
- $1 + 2 = 3$  (**true**)
- Protons have no electric charge (**false**)
- $13 > 37$  (**false**)

## Operators: and, or

Propositions can be grouped together with **operators** such as **and, or**.



## Operators: and, or

Propositions can be grouped together with **operators** such as **and**, **or**.

- The **and** operator returns a **true** statement only if **both** the statements it groups are themselves **true**, otherwise it returns **false**.

## Operators: and, or

Propositions can be grouped together with **operators** such as **and**, **or**.

- The **and** operator returns a **true** statement only if **both** the statements it groups are themselves **true**, otherwise it returns **false**.
- The **or** operator returns **true** if **at least** one of the statements it groups is true.

## Example

$1 + 2 = 3$  and  $3 - 5 = -2 \Rightarrow \text{true}$

## Example

$1 + 2 = 3$  and  $3 - 5 = -2$   $\Rightarrow$  true

$1 + 2 = 3$  and  $2 \times 4 = 7$   $\Rightarrow$  false

## Example

$$1 + 2 = 3 \text{ and } 3 - 5 = -2 \Rightarrow \text{true}$$

$$1 + 2 = 3 \text{ and } 2 \times 4 = 7 \Rightarrow \text{false}$$

$$\frac{10}{2} = 1 \text{ and } 2^4 = 16 \Rightarrow \text{false}$$

## Example

$$1 + 2 = 3 \text{ and } 3 - 5 = -2 \Rightarrow \text{true}$$

$$1 + 2 = 3 \text{ and } 2 \times 4 = 7 \Rightarrow \text{false}$$

$$\frac{10}{2} = 1 \text{ and } 2^4 = 16 \Rightarrow \text{false}$$

$$7 < 5 \text{ and } 10 + 2 = 13 \Rightarrow \text{false}$$

### Example

$1 + 2 = 3$  or  $3 > 7 \Rightarrow \text{true}$

## Example

$1 + 2 = 3$  or  $3 > 7$   $\Rightarrow$  true

$0 + 3 = -1$  or  $1 = 1$   $\Rightarrow$  true



## Example

$1 + 2 = 3$  or  $3 > 7$   $\Rightarrow$  true

$0 + 3 = -1$  or  $1 = 1$   $\Rightarrow$  true

$2 \times 2 = 4$  or  $2 + 0 = 2$   $\Rightarrow$  true

## Example

$$1 + 2 = 3 \text{ or } 3 > 7 \Rightarrow \text{true}$$

$$0 + 3 = -1 \text{ or } 1 = 1 \Rightarrow \text{true}$$

$$2 \times 2 = 4 \text{ or } 2 + 0 = 2 \Rightarrow \text{true}$$

$$3 \times 7 = 10 \text{ or } \frac{1}{2} < \frac{1}{10} \Rightarrow \text{false}$$

# Operators: Truth Table

We can summarize the behaviour of operators in a **truth table** :

<i>A</i>	<i>B</i>	AND	OR
true	true	true	true
true	false	false	true
false	true	false	true
false	false	false	false

# Mathematical Notation

Other **notations** that will be used throughout this course:

Symbol	In words
$\neg a$	<b>not</b> $a$
$a \wedge b$	$a$ <b>and</b> $b$
$a \vee b$	$a$ <b>or</b> $b$
$a \Rightarrow b$	$a$ <b>implies</b> $b$
$a \Leftrightarrow b$	$a$ <b>is equivalent to</b> $b$
$\forall x$	<b>For all</b> $x$ (...)
$\exists x$	<b>There exists</b> $x$ such that (...)
$a := b$	$a$ <b>is defined to be</b> $b$

## Definition

A **set** is a collection of **elements**. Elements of a set can be any concept - be it physical (a chair, a car, a tapir) or abstract (a number, an idea).

## Definition

A **set** is a collection of **elements**. Elements of a set can be any concept - be it physical (a chair, a car, a tapir) or abstract (a number, an idea).

## Note

In this course we consider only **mathematical objects** as elements of sets.

## Definition

A **set** is a collection of **elements**. Elements of a set can be any concept - be it physical (a chair, a car, a tapir) or abstract (a number, an idea).

## Note

In this course we consider only **mathematical objects** as elements of sets.

Sets can have a **finite** or **infinite** number of elements.

Sets are denoted with curly brackets.



Sets are denoted with curly brackets.

## Example

$$\{1, 2, 3, 4\}, \quad \left\{-4, \frac{3}{7}, 0, \pi, i, 0.1\right\}, \quad \{\text{all even numbers}\}.$$

The order of elements in a set **does not matter**.

The order of elements in a set **does not matter**.

## Example

The following sets are all identical:

$$\{1, 2, 3, 4\} = \{1, 3, 2, 4\} = \{2, 1, 4, 3\}.$$

The order of elements in a set **does not matter**.

## Example

The following sets are all identical:

$$\{1, 2, 3, 4\} = \{1, 3, 2, 4\} = \{2, 1, 4, 3\}.$$

## Note

There is no repetition in sets, i.e.  $\{1, 1, 3, 3, 3, 3, 5\}$  is not a proper set, contrary to e.g.  $\{1, 3, 5\}$ .

Sets can be denoted as conditions, using a vertical separator to denote **conditions**.

Sets can be denoted as conditions, using a vertical separator to denote **conditions**.

## Example

The following set contains all odd numbers between 0 and 10:

$$\{x \text{ is odd} \mid 0 < x < 10\}.$$

Sets can be denoted as conditions, using a vertical separator to denote **conditions**.

## Example

The following set contains all odd numbers between 0 and 10:

$$\{x \text{ is odd} \mid 0 < x < 10\}.$$

It can also be written explicitly:

$$\{1, 3, 5, 7, 9\}.$$

Sets are usually denoted with an uppercase latin letter, while their elements as lowercase latin or greek letters. The notation  $\in$  means that an element belongs to a set.



Sets are usually denoted with an uppercase latin letter, while their elements as lowercase latin or greek letters. The notation  $\in$  means that an element belongs to a set.

## Example

For the two sets

$$A = \{1, 2, 5, 7\}, \quad B = \{\text{even numbers}\},$$

all the following propositions are true:

$$1 \in A, \quad 2 \in A, \quad 5 \in A, \quad 7 \in A,$$

$$2 \in B, \quad 1 \notin B, \quad 5 \notin B, \quad 7 \notin B.$$

The number of elements in a set (also called its **cardinality**) is denoted with two vertical bars.

The number of elements in a set (also called its **cardinality**) is denoted with two vertical bars.

## Example

$$S = \{-3, 0, -2, 7, 1\} \Rightarrow |S| = 5.$$

# The Empty Set

An important special set is the **empty set**, which is the set containing no elements. It is denoted by  $\emptyset$ , and has the unique property that

$$|\emptyset| = 0.$$

## Subsets and Supersets

If a set  $A$  contains all the elements in a set  $B$  (and perhaps additional elements), then  $B$  is said to be a **subset** of  $A$ , and  $A$  a **superset** of  $B$ .

## Subsets and Supersets

If a set  $A$  contains all the elements in a set  $B$  (and perhaps additional elements), then  $B$  is said to be a **subset** of  $A$ , and  $A$  a **superset** of  $B$ .

### Example

The sets

$$A = \{0, -3\}, \quad B = \{5, -3, 1\}, \quad C = \{-2, 2, 1\},$$

are some of the subsets of

$$D = \{0, -3, 5, 1, 2, -2\}.$$

Equivalently,  $D$  is a superset of  $A, B$  and  $C$ .

### Note

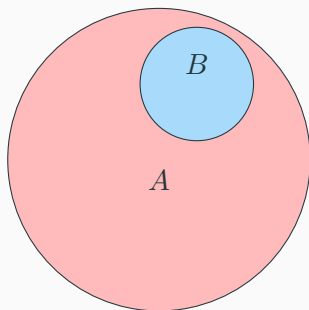
All sets are supersets and subsets of themselves. This is a direct consequence of the definition of supersets and subsets.

## Subsets and Supersets

We denote that  $A$  is a superset of  $B$  as

$$B \subseteq A.$$

A **Venn Diagram** representation of this fact looks as following:





If for some two sets  $A, B$  both  $A \subseteq B$  **and**  $B \subseteq A$ , then the sets are identical.

Formally, this fact is written as

$$A \subseteq B \wedge B \subseteq A \Leftrightarrow A = B.$$

# Intersections and Unions

## Definition

The **intersection** of two sets  $A$  and  $B$  is the set of all elements that are **both** in  $A$  and in  $B$ .

## Definition

The **intersection** of two sets  $A$  and  $B$  is the set of all elements that are **both** in  $A$  and in  $B$ .

## Example

Given the sets

$$A = \{1, 2, 5, 6, 7\}, \quad B = \{-1, 0, 1, 5, 10, 13, 15\},$$

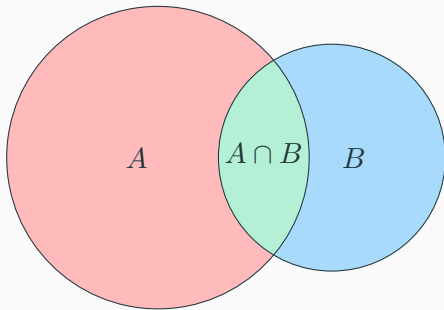
the intersection of  $A$  and  $B$  is  $\{1, 5\}$ .

The symbol denoting intersection is  $\cap$ . An intersection can be formally defined as

$$A \cap B = \{x \mid x \in A \wedge x \in B\}$$

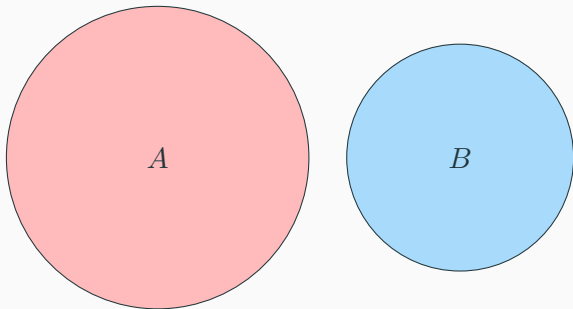
(read: "the intersection of  $A$  and  $B$  is the set containing all elements  $x$ , such that  $x$  is in  $A$  and  $x$  is in  $B$ ")

A Venn diagram visualization of  $A \cap B$  (green area):



## Intersections and Unions

If the intersection of two sets is empty ( $A \cap B = \emptyset$ ), then the sets are said to be **disjoint** :



## Definition

The **union** of two sets  $A, B$  is the set of all elements that are either in  $A$  or in  $B$  (or both).

### Definition

The **union** of two sets  $A, B$  is the set of all elements that are either in  $A$  or in  $B$  (or both).

### Example

The union of the sets

$$A = \{-5, 7, 1\}, B = \{10, -2, -5, 2\},$$

is

$$A \cup B = \{10, -2, -5, 2, 7, 1\}.$$

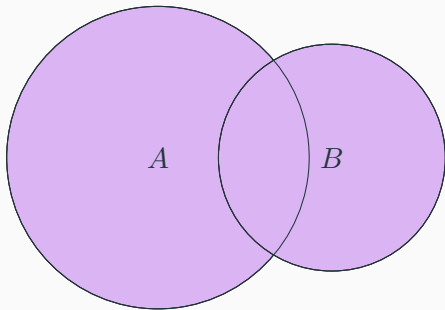


The symbol denoting union is  $\cup$ . A union can be formally defined as

$$A \cup B = \{x \mid x \in A \vee x \in B\}$$

(read: "the union of  $A$  and  $B$  is the set containing all elements  $x$ , such that  $x$  is in  $A$  or  $x$  is in  $B$ ")

A Venn diagram visualization of  $A \cup B$  (purple area):



The number of elements in a union of two sets  $A$  and  $B$  is

$$|A \cup B| = |A| + |B| - |A \cap B|$$

The number of elements in a union of two sets  $A$  and  $B$  is

$$|A \cup B| = |A| + |B| - |A \cap B|$$

### Note

If  $A, B$  are disjoint,  $|A \cup B| = |A| + |B|$  (because  $|A \cap B| = 0$ ).

## Definition

The **difference** of  $A$  and  $B$  is the set of all elements in  $A$  that **are not** elements of  $B$ . This is written as  $A - B$  (or sometimes  $A \setminus B$ ).

# Difference of Sets

## Definition

The **difference** of  $A$  and  $B$  is the set of all elements in  $A$  that **are not** elements of  $B$ . This is written as  $A - B$  (or sometimes  $A \setminus B$ ).

## Example

For the sets

$$A = \{1, 5, 9, 10\}, \quad B = \{-3, 2, 5, 9, 13\},$$

The differences are

$$A - B = \{1, 10\}, \quad B - A = \{-3, 2, 13\}.$$

## Difference of Sets

Formally:

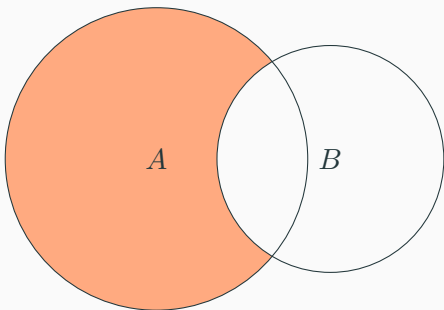
$$A - B = \{x \mid x \in A, x \notin B\}$$

## Difference of Sets

Formally:

$$A - B = \{x \mid x \in A, x \notin B\}$$

A Venn diagram visualization of  $A - B$  (orange area):





## Definition

The **complement** of a set  $A$  in relation to a superset  $Z \supset A$  is the difference  $Z - A$ , and is denoted  $A^c$ .

## Definition

The **complement** of a set  $A$  in relation to a superset  $Z \supset A$  is the difference  $Z - A$ , and is denoted  $A^c$ .

## Example

For the sets

$$Z = \{1, 2, 3, 4, 5\}, \quad A = \{1, 2, 3\},$$

The complement of  $A$  in relation to  $Z$  is

$$A^c = \{4, 5\}$$

# Complement

Formally:

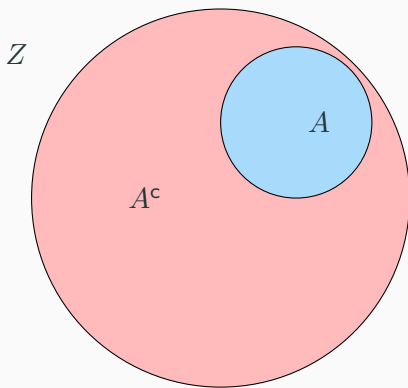
$$A^c = \{x \in Z \mid x \notin A\}.$$

# Complement

Formally:

$$A^c = \{x \in Z \mid x \notin A\}.$$

A Venn diagram representation:



## Definition

The set of all subsets of a given set  $A$  is called the **powerset of  $A$**  .

## Definition

The set of all subsets of a given set  $A$  is called the **powerset of  $A$**  .

## Example

All the subsets of  $A = \{1, 2, 3\}$  are:

$$\emptyset, \{1\}, \{2\}, \{3\}, \{1, 2\}, \{1, 3\}, \{2, 3\}, \{1, 2, 3\} .$$

Thus, the power set of  $A$  is

$$P(A) = \{\emptyset, \{1\}, \{2\}, \{3\}, \{1, 2\}, \{1, 3\}, \{2, 3\}, \{1, 2, 3\}\} .$$

## Note

The empty set  $\emptyset$  is a subset of all sets. Each set is also a subset of itself.

## Important Number Sets

Some important number sets, which will be used frequently in the course (all with infinite number of elements):



## Important Number Sets

Some important number sets, which will be used frequently in the course (all with infinite number of elements):

- **The natural numbers** (symbol:  $\mathbb{N}$ ). These are the numbers  $1, 2, 3, \dots$

## Important Number Sets

Some important number sets, which will be used frequently in the course (all with infinite number of elements):

- **The natural numbers** (symbol:  $\mathbb{N}$ ). These are the numbers  $1, 2, 3, \dots$
- **The integers** (symbol:  $\mathbb{Z}$ ). These are the "whole numbers" (i.e. not fractions). They include all the natural numbers together with their negatives (i.e.  $-1, -2, -3, \dots$ ) and 0.

## Important Number Sets

Some important number sets, which will be used frequently in the course (all with infinite number of elements):

- **The natural numbers** (symbol:  $\mathbb{N}$ ). These are the numbers  $1, 2, 3, \dots$
- **The integers** (symbol:  $\mathbb{Z}$ ). These are the "whole numbers" (i.e. not fractions). They include all the natural numbers together with their negatives (i.e.  $-1, -2, -3, \dots$ ) and 0.
- **The rational numbers** (symbol:  $\mathbb{Q}$ ). As their name suggests, they are ratios between two integers (e.g.  $\frac{1}{2}, \frac{-5}{3}, \frac{7}{13}$ ).

## Important Number Sets

Some important number sets, which will be used frequently in the course (all with infinite number of elements):

- **The natural numbers** (symbol:  $\mathbb{N}$ ). These are the numbers  $1, 2, 3, \dots$
- **The integers** (symbol:  $\mathbb{Z}$ ). These are the "whole numbers" (i.e. not fractions). They include all the natural numbers together with their negatives (i.e.  $-1, -2, -3, \dots$ ) and 0.
- **The rational numbers** (symbol:  $\mathbb{Q}$ ). As their name suggests, they are ratios between two integers (e.g.  $\frac{1}{2}, \frac{-5}{3}, \frac{7}{13}$ ).
- **The real numbers** (symbol:  $\mathbb{R}$ ). These are all the numbers on the number line (e.g.  $2, \pi, \frac{\sqrt{3}}{17}, \sqrt{5}, -7.2, e^\pi$ ). A proper definition of the real numbers is beyond the scope of this course.

Additionally, the **Complex Numbers** are the set of all numbers

$$z = a + bi,$$

where  $a$  and  $b$  are both real numbers, and  $i$  is the imaginary unit, i.e.  $i = \sqrt{-1}$ .

The complex number set has the notation  $\mathbb{C}$ .

# Important Number Sets

Table summary:

Symbol	Name	Definition
$\mathbb{N}$	Natural numbers	$\{1, 2, 3, 4, \dots\}$
$\mathbb{Z}$	Integers	$\{0, \pm x \mid x \in \mathbb{N}\}$
$\mathbb{Q}$	Rational numbers	$\left\{\frac{p}{q} \mid p \in \mathbb{Z}, q \in \mathbb{N}\right\}$
$\mathbb{R}$	Real numbers	Not in this course
$\mathbb{C}$	Complex numbers	$\{a + ib \mid a, b \in \mathbb{R}, i = \sqrt{-1}\}$

### Note

The relations between these sets are

$$\mathbb{N} \subset \mathbb{Z} \subset \mathbb{Q} \subset \mathbb{R} \subset \mathbb{C}$$

(the symbol  $\subset$  means "a proper subset")

## Important Number Sets

### Note

The relations between these sets are

$$\mathbb{N} \subset \mathbb{Z} \subset \mathbb{Q} \subset \mathbb{R} \subset \mathbb{C}$$

(the symbol  $\subset$  means "a proper subset")

### Note

Although each of these sets is infinite, the actual number of elements in  $\mathbb{R}$  and  $\mathbb{C}$  **is bigger** than the number of elements in  $\mathbb{N}$ ,  $\mathbb{Z}$  and  $\mathbb{Q}$ . There are different kinds of infinities.



The **interval**  $[a, b]$  is the subset of  $\mathbb{R}$  defined as

$$[a, b] = \{x \in \mathbb{R} \mid a \leq x \leq b\}.$$

## Example

The interval  $I = [-5, 3]$  is the set of all real numbers that are **greater than or equal** to  $-5$  and are **smaller than or equal**  $3$ .

Some examples:

$$-5.1 \notin I, -5 \in I, 0 \in I, 2 \in I, 3 \in I, 4 \notin I.$$

The interval  $(a, b)$  is the subset of  $\mathbb{R}$  defined as

$$(a, b) = \{x \in \mathbb{R} \mid a < x < b\}.$$

(i.e. same as  $[a, b]$  but excluding the actual values  $a$  and  $b$ )

## Example

The interval  $I = (-5, 3)$  is the set of all real numbers that are **greater than**  $-5$  and are **smaller than**  $3$ .

Some examples:

$$-5.1 \notin I, -5 \notin I, 0 \in I, 2 \in I, 3 \notin I, 4 \notin I.$$

Similarly, the interval  $[a, b)$  is the subset of  $\mathbb{R}$  defined as

$$[a, b) = \{x \in \mathbb{R} \mid a \leq x < b\},$$

and the interval  $(a, b]$  is the subset of  $\mathbb{R}$  defined as

$$(a, b] = \{x \in \mathbb{R} \mid a < x \leq b\}.$$

(i.e. in the notation for intervals a square bracket means "less/more than or equal to", while a round bracket means "less/more than" - without the "equal to" part)

## Definition

The **cartesian product** of two sets  $A, B$  (denoted  $A \times B$ ) is the set of all possible **ordered** pairs, where the first component is an element of  $A$  and the second component is an element of  $B$ .

# Cartesian Products

## Definition

The **cartesian product** of two sets  $A, B$  (denoted  $A \times B$ ) is the set of all possible **ordered** pairs, where the first component is an element of  $A$  and the second component is an element of  $B$ .

## Example

Consider  $A = \{1, 2, 3\}$ ,  $B = \{x, y\}$ . Then:

$$A \times B = \{(1, x), (1, y), (2, x), (2, y), (3, x), (3, y)\}$$

## Note

The cartesian product of two sets  $A, B$  is not commutative, i.e.

$$A \times B \neq B \times A,$$

unless  $A = B$  or any one of the sets (or both) is the empty set.

# Cartesian Products

Defining a cartesian product formally:

$$A \times B = \{(a, b) \mid a \in A, b \in B\}.$$

## Cartesian Products

Defining a cartesian product formally:

$$A \times B = \{(a, b) \mid a \in A, b \in B\}.$$

The number of elements in a cartesian product is

$$|A \times B| = |A| \cdot |B|.$$



## Cartesian Products

Defining a cartesian product formally:

$$A \times B = \{(a, b) \mid a \in A, b \in B\}.$$

The number of elements in a cartesian product is

$$|A \times B| = |A| \cdot |B|.$$

The definition of a cartesian product can be expanded to  $n \in \mathbb{N}$  sets  $A_1, A_2, \dots, A_n$ :

$$A_1 \times A_2 \times \cdots \times A_n = \{(a_1, a_2, \dots, a_n) \mid a_1 \in A_1, a_2 \in A_2, \dots, a_n \in A_n\}$$

The definition can be made more compact by the use of the product symbol  $\prod$ :

$$\prod_{i=1}^n A_i = \{(a_1, a_2, \dots, a_i) \mid a_i \in A_i, i = 1, 2, \dots, n\}.$$

# Cartesian Products

The definition can be made more compact by the use of the product symbol  $\prod$ :

$$\prod_{i=1}^n A_i = \{(a_1, a_2, \dots, a_i) \mid a_i \in A_i, i = 1, 2, \dots, n\}.$$

## Note

The symbol  $\prod$  is a generalized product notation. It will be discussed in more details later in the course.

## Cartesian Products

A cartesian product of the same set is written in an similar way to a power. For example

$$\mathbb{R} \times \mathbb{R} = \mathbb{R}^2,$$

$$\mathbb{R} \times \mathbb{R} \times \mathbb{R} = \mathbb{R}^3.$$

These are, respectively, sets of pairs of real numbers, e.g.

$(-3, 1)$ ,  $(\pi, 2)$ ,  $(-\frac{\sqrt{7}}{13}, 0)$ , and triples of real numbers, e.g.

$(1, 2, -\pi)$ ,  $(-6, \frac{1}{\sqrt{\pi}}, 0.2)$ ,  $(\frac{1}{51}, \sqrt{3}, -4)$ .

## Example

For the set  $A = \{a, b\}$ ,

$$A^3 = \{(aaa), (aab), (aba), (abb), (baa), (bab), (bba), (bbb)\}.$$

## Example

For the set  $A = \{a, b\}$ ,

$$A^3 = \{(aaa), (aab), (aba), (abb), (baa), (bab), (bba), (bbb)\}.$$

For the set  $B = \{1, 2, 3\}$ ,

$$B^2 = \{(1, 1), (1, 2), (1, 3), (2, 1), (2, 2), (2, 3), \\ (3, 1), (3, 2), (3, 3)\}.$$

### Definition

A **relation** between two sets  $A$  and  $B$  is a way to "connect" the elements in the two sets in pairs. It is a subset of the cartesian product  $A \times B$ .

## Definition

A **relation** between two sets  $A$  and  $B$  is a way to "connect" the elements in the two sets in pairs. It is a subset of the cartesian product  $A \times B$ .

## Example

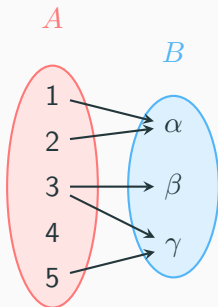
An example relation between the sets  $A = \{1, 2, 3, 4, 5\}$  and  $B = \{\alpha, \beta, \gamma\}$  is

$$R = \{(1, \alpha), (2, \alpha), (3, \beta), (3, \gamma), (5, \gamma)\}.$$



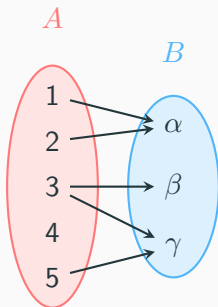
## Relations Between Sets

The previous relation can be visually represented as following:



## Relations Between Sets

The previous relation can be visually represented as following:



### Note

Notice how not all elements are connected, and some elements in each set are connected to the same element in the other set.

## Reversed Relations

The previous relation can be reversed, yielding a subset of  $B \times A$ :

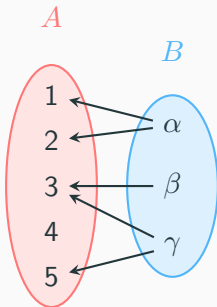
$$R^{-1} = \{(\alpha, 1), (\alpha, 2), (\beta, 3), (\gamma, 3), (\gamma, 5)\}.$$

## Reversed Relations

The previous relation can be reversed, yielding a subset of  $B \times A$ :

$$R^{-1} = \{(\alpha, 1), (\alpha, 2), (\beta, 3), (\gamma, 3), (\gamma, 5)\}.$$

Graphically:



## Definition

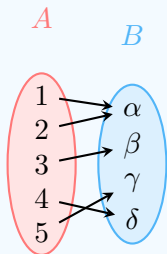
A **function** between the sets  $A, B$  is a relation in which for every element  $a \in A$  there is exactly **one** connection to an element  $b \in B$ .

## Definition

A **function** between the sets  $A, B$  is a relation in which for every element  $a \in A$  there is exactly **one** connection to an element  $b \in B$ .

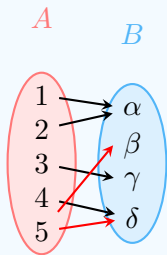
## Example

A function from a set  $A$  to a set  $B$ :



## Example

A relation which is **NOT** a function from  $A$  to  $B$ :



Two additional terms that are used interchangeably with function are **transformation** and **map**.



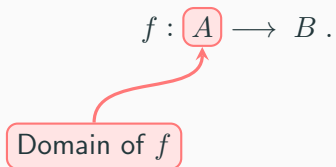
## Note

A function can have more than one element  $a \in A$  connected to the same element  $b \in B$ . The only restriction is that no element  $a \in A$  is connected to **more than one** element  $b \in B$ .

A common notation to a function  $f$  connecting between elements of the sets  $A$  and  $B$  is

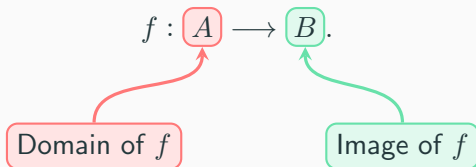
$$f: A \longrightarrow B .$$

A common notation to a function  $f$  connecting between elements of the sets  $A$  and  $B$  is

$$f : A \longrightarrow B .$$


A diagram illustrating the domain of the function  $f$ . The letter  $A$  in the expression  $f : A \longrightarrow B .$  is enclosed in a red rounded square. A red arrow originates from a red rounded rectangle containing the text "Domain of  $f$ " and points to the  $A$ .

A common notation to a function  $f$  connecting between elements of the sets  $A$  and  $B$  is



When used in practice, a common notation to show that an element  $x \in A$  is connected to another element  $y \in B$  is

$$f(x) = y,$$

i.e. the function  $f$  applied to the element  $x \in A$  returns the element  $y \in B$ .

In part 3 of the course we will deal with functions of the form

$$f : \mathbb{R} \longrightarrow \mathbb{R},$$

which we call **real functions**, i.e. functions that take a real number  $x$  and return a real number  $y$ .

In part 3 of the course we will deal with functions of the form

$$f : \mathbb{R} \longrightarrow \mathbb{R},$$

which we call **real functions**, i.e. functions that take a real number  $x$  and return a real number  $y$ .

## Example

The functions

$$f_1(x) = 2x^2 - 5, \quad f_2(x) = \sin\left(\frac{x}{3}\right), \quad f_3(x) = \frac{1}{\sqrt{2\pi}}e^{-\frac{x^2}{2}}$$

are all real functions.

## Plotting Real Functions

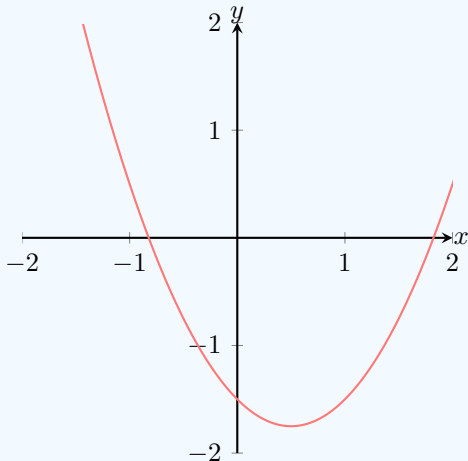
We can plot a real function  $f$  on a cartesian coordinate system by drawing a dot in each coordinate  $(x, y)$ , where  $x$  is an element in the domain of  $f$ , and  $y$  is its image (i.e.  $f(x) = y$ ).



# Plotting Real Functions

## Example

Plotting the function  $f(x) = x^2 - x - 1.5$ :



## Injective, Surjective and Bijective Functions

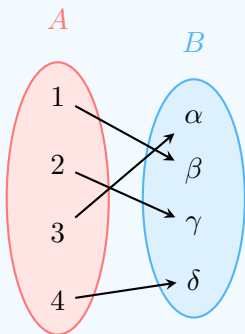
A function is called **injective** if each of the elements in its **image** is connected to by a single element in its **domain**.

# Injective, Surjective and Bijective Functions

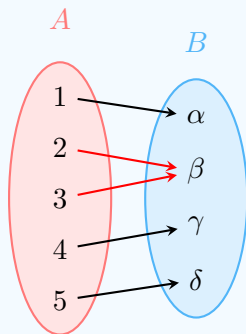
A function is called **injective** if each of the elements in its **image** is connected to by a single element in its **domain**.

## Example

Injective



Not injective



## Injective, Surjective and Bijective Functions

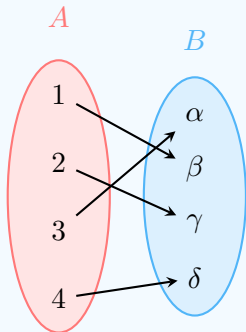
A function is called **surjective** if all of the elements in its **image** are connected to by some element in its **domain**.

# Injective, Surjective and Bijective Functions

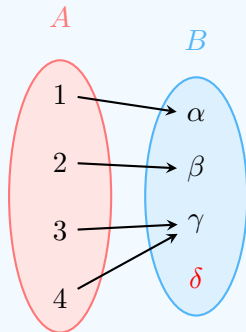
A function is called **surjective** if all of the elements in its **image** are connected to by some element in its **domain**.

## Example

### Surjective



### Not surjective



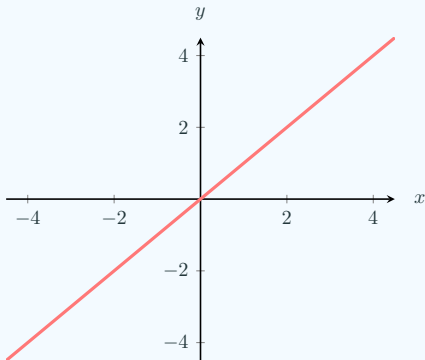
A function that is both **injective** and **surjective** is called **bijective**.

# Injective, Surjective and Bijective Functions

## Example

Let's look at a few examples of real injective, surjective and bijective functions over  $\mathbb{R}$ :

- $f(x) = x$ , injective + surjective = bijective.

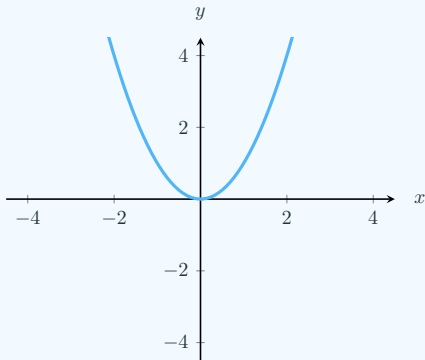


# Injective, Surjective and Bijective Functions

## Example

Let's look at a few examples of real injective, surjective and bijective functions over  $\mathbb{R}$ :

- $f(x) = x^2$ , neither injective nor surjective.



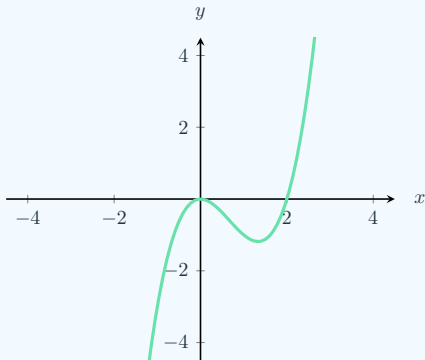


# Injective, Surjective and Bijective Functions

## Example

Let's look at a few examples of real injective, surjective and bijective functions over  $\mathbb{R}$ :

- $f(x) = x^3 - 2x^2$ , surjective.

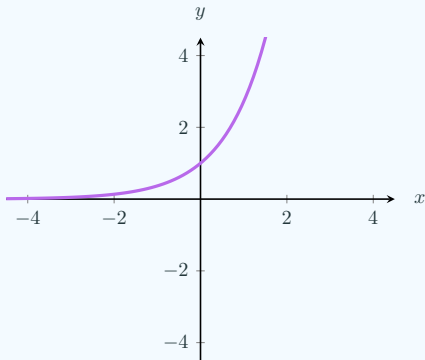


# Injective, Surjective and Bijective Functions

## Example

Let's look at a few examples of real injective, surjective and bijective functions over  $\mathbb{R}$ :

- $f(x) = e^x$ , injective.

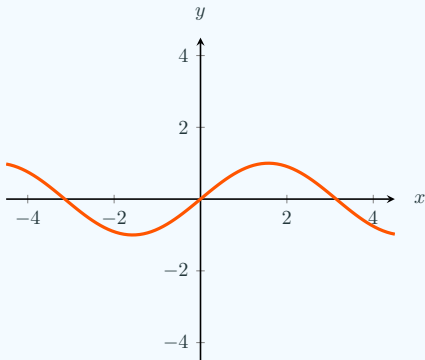


# Injective, Surjective and Bijective Functions

## Example

Let's look at a few examples of real injective, surjective and bijective functions over  $\mathbb{R}$ :

- $f(x) = \sin(x)$ , neither injective nor surjective.



## Note

Every non-surjective function can be made surjective by excluding the elements its image that are not connected to by any element in its domain.

For example, the function  $f(x) = \sin(x)$  is not surjective as a function  $f : \mathbb{R} \rightarrow \mathbb{R}$ , but is surjective as a function  $f : \mathbb{R} \rightarrow [-1, 1]$ .

# Multivariable Functions

Functions may have several arguments and return several arguments.

# Multivariable Functions

Functions may have several arguments and return several arguments.

## Example

The following functions take as input three real numbers, and return a single real number ( $f : \mathbb{R}^3 \rightarrow \mathbb{R}$ ). The return value of some functions for a triplet of real numbers,  $(-5, 7, 1)$ , are:

- $f(x, y, z) = x + y + z \Rightarrow f(-5, 7, 1) = -5 + 7 + 1 = 3$

# Multivariable Functions

Functions may have several arguments and return several arguments.

## Example

The following functions take as input three real numbers, and return a single real number ( $f : \mathbb{R}^3 \rightarrow \mathbb{R}$ ). The return value of some functions for a triplet of real numbers,  $(-5, 7, 1)$ , are:

- $f(x, y, z) = x + y + z \Rightarrow f(-5, 7, 1) = -5 + 7 + 1 = 3$
- $f(x, y, z) = x^2 - y^2 \Rightarrow f(-5, 7, 1) = 25 - 49 = -24$

# Multivariable Functions

Functions may have several arguments and return several arguments.

## Example

The following functions take as input three real numbers, and return a single real number ( $f : \mathbb{R}^3 \rightarrow \mathbb{R}$ ). The return value of some functions for a triplet of real numbers,  $(-5, 7, 1)$ , are:

- $f(x, y, z) = x + y + z \Rightarrow f(-5, 7, 1) = -5 + 7 + 1 = 3$
- $f(x, y, z) = x^2 - y^2 \Rightarrow f(-5, 7, 1) = 25 - 49 = -24$
- $f(x, y, z) = \frac{x}{\sqrt{y+z}} \Rightarrow f(-5, 7, 1) = \frac{5}{\sqrt{7+1}}$



## Example

The function  $f : \mathbb{Z} \times \mathbb{N} \longrightarrow \mathbb{Q}$  is defined as

$$f(p, q) = \frac{p}{q}.$$

The return values of  $f$  for some example inputs are

## Example

The function  $f : \mathbb{Z} \times \mathbb{N} \longrightarrow \mathbb{Q}$  is defined as

$$f(p, q) = \frac{p}{q}.$$

The return values of  $f$  for some example inputs are

- $f(1, 2) = \frac{1}{2}$ ,

## Example

The function  $f : \mathbb{Z} \times \mathbb{N} \longrightarrow \mathbb{Q}$  is defined as

$$f(p, q) = \frac{p}{q}.$$

The return values of  $f$  for some example inputs are

- $f(1, 2) = \frac{1}{2}$ ,
- $f(-5, 2) = -\frac{5}{2} = -2.5$ ,

## Example

The function  $f : \mathbb{Z} \times \mathbb{N} \longrightarrow \mathbb{Q}$  is defined as

$$f(p, q) = \frac{p}{q}.$$

The return values of  $f$  for some example inputs are

- $f(1, 2) = \frac{1}{2}$ ,
- $f(-5, 2) = -\frac{5}{2} = -2.5$ ,
- $f(0, 13) = \frac{0}{13} = 0$ .

# Composition of Functions

Functions can be **composed** together, generating new functions.

# Composition of Functions

Functions can be **composed** together, generating new functions.

## Example

Consider the functions

$$f(x) = x^2, \quad g(x) = \sin(x).$$

We can compose the two functions in two ways:

# Composition of Functions

Functions can be **composed** together, generating new functions.

## Example

Consider the functions

$$f(x) = x^2, \quad g(x) = \sin(x).$$

We can compose the two functions in two ways:

- $g_1(x) = f(g(x)) = [\sin(x)]^2$ , and

# Composition of Functions

Functions can be **composed** together, generating new functions.

## Example

Consider the functions

$$f(x) = x^2, \quad g(x) = \sin(x).$$

We can compose the two functions in two ways:

- $g_1(x) = f(g(x)) = [\sin(x)]^2$ , and
- $g_2(x) = g(f(x)) = \sin(x^2)$ .



## Composition of Functions

We denote a composition of two functions  $f : A \rightarrow B$  and  $g : B \rightarrow C$  as

$$g \circ f : A \rightarrow C.$$

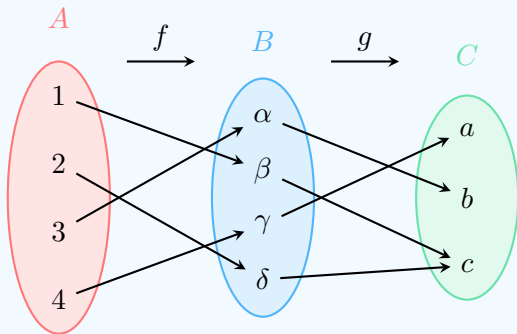
### Note

For a composition to be valid, the **domain** of the second function (here  $g$ ) must be the same as the **image** of the first function.

# Composition of Functions

## Example

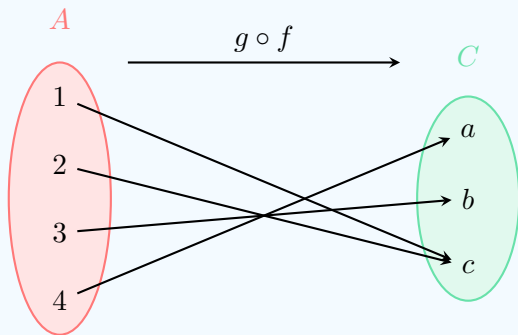
A graphical representation of composing two functions:



# Composition of Functions

## Example

A graphical representation of composing two functions:



## Definition

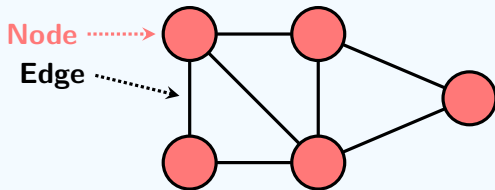
A **graph** is a mathematical structure composed of **nodes** connected to other nodes by **edges** .

## Definition

A **graph** is a mathematical structure composed of **nodes** connected to other nodes by **edges**.

## Example

A graph with 5 nodes and 7 edges:



# Graphs

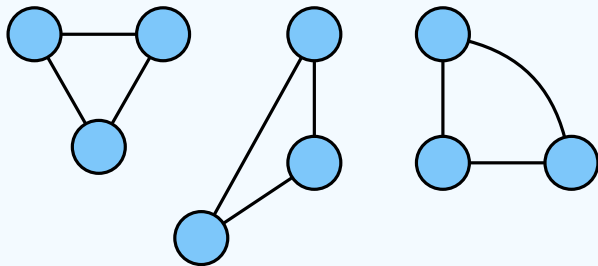
In the graphical representation of a graph, the actual position of nodes does not matter - what matters are the connections (edges) between them.

# Graphs

In the graphical representation of a graph, the actual position of nodes does not matter - what matters are the connections (edges) between them.

## Example

The following three graphs are identical:



## Definition

A graph in which edges have directions is called a **directed graph**.

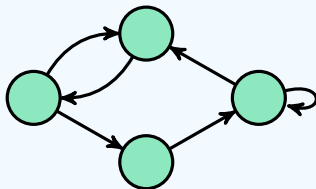


## Definition

A graph in which edges have directions is called a **directed graph**.

## Example

A directed graph with 4 nodes and 6 edges:



## Definition

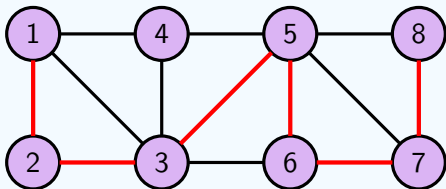
A **path** in a graph is a sequence of edges in which each edge shares a vertex with the previous edge (except the first edge).

## Definition

A **path** in a graph is a sequence of edges in which each edge shares a vertex with the previous edge (except the first edge).

## Example

A path in a graph (note that the nodes are labeled):



## Definition

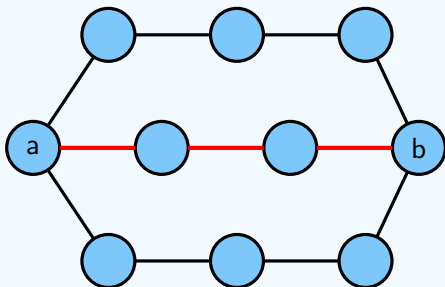
When the start and end vertices coincide the path is known as a **circle** . A directed circle is known as a **cycle** .

## Definition

If one or more paths exist between two vertices  $a, b$  in a graph, the number of edges in the shortest path is defined to be the **distance** between the two vertices, and is denoted as  $\text{dist}(a, b)$ .

## Example

In the following graph three paths between vertices  $a$  and  $b$  are shown. The number of edges in the shortest path, highlighted in red, is defined as the distance  $\text{dist}(a, b)$ , and is equal to 3.



## Definition

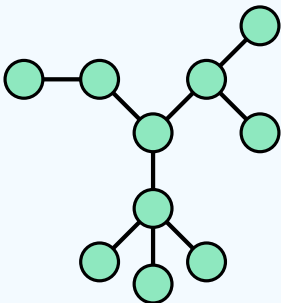
A **tree** is a graph with no circles.

## Definition

A **tree** is a graph with no circles.

## Example

A tree (notice that no circles are present):

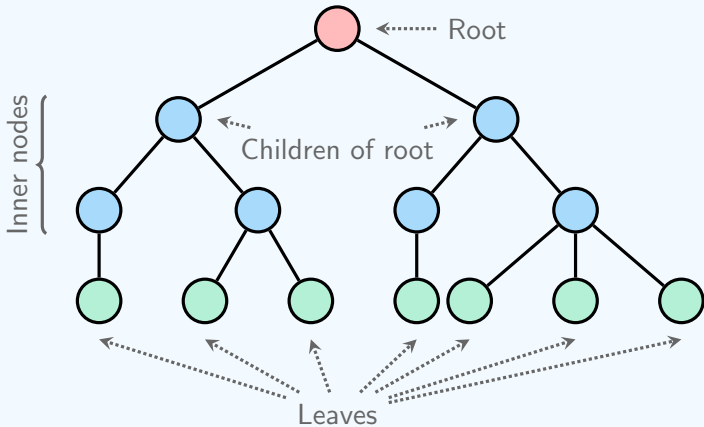




Some trees have a distinctive **root** node, and are known as **rooted trees**. A node that is "branched" from a higher level node is called a **child node**. The last level nodes are called **leaves** (singular: leaf). The rest of the nodes are known as **inner nodes**.

## Example

A rooted tree, with the root node highlighted in red and the leaves in green:

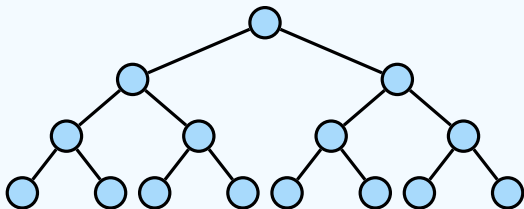


## Definition

A tree with 2 children per node (except the leaves) is called a **binary tree**. Similarly, trees can be ternary, quaternary, etc.

## Example

A binary tree:



Rooted trees are used to describe hierarchies, e.g. in biological systematics, organisations or nested directories of data.

## Definition

The **complete graph**  $K_n$  is the graph with  $n$  vertices where every pair of different vertices is connected by an edge (Also called a **clique** ).

## Definition

The **complete graph**  $K_n$  is the graph with  $n$  vertices where every pair of different vertices is connected by an edge (Also called a **clique**).

## Example

The cliques  $K_1, \dots, K_6$ :



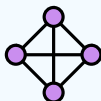
$K_1$



$K_2$



$K_3$



$K_4$



$K_5$



$K_6$