GEORG-AUGUST-UNIVERSITÄT
GÖTTINGEN

Mathematics and Computer Science (B.MES.108)
Summer Semester, 2020
Part 1: Linear Algebra for Non-Mathematicians

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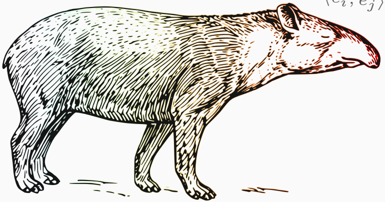
$(AB)^{\top} = B^{\top} A^{\top}$ $\quad \mathbb{R}^n \xrightarrow{T} \mathbb{R}^m$

$\vec{v} = \sum_{i=1}^{n} \alpha_i \hat{e}_i$ $\quad A = Q \Lambda Q^{-1}$

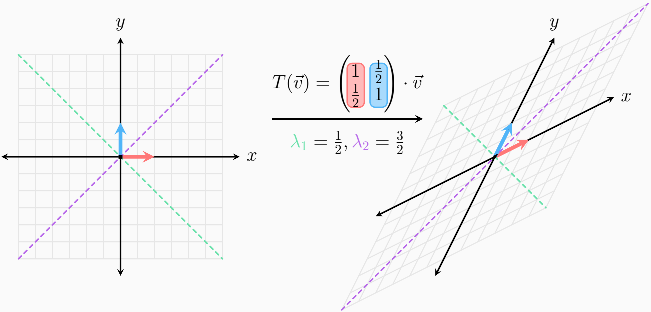
$\text{Rot}(\theta) = \begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix}$ $\quad A\vec{v} = \lambda \vec{v}$

$T(\alpha \vec{u} + \beta \vec{v}) = \alpha T(\vec{u}) + \beta T(\vec{v})$

$\langle \hat{e}_i, \hat{e}_j \rangle = \delta_{ij}$



Chapter 6: Eigenvectors and Eigenvalues



Definition

An **eigenvector** of a transformation $T : \mathbb{R}^n \to \mathbb{R}^m$ is a vector that doesn't change its direction under the transformation.

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Example

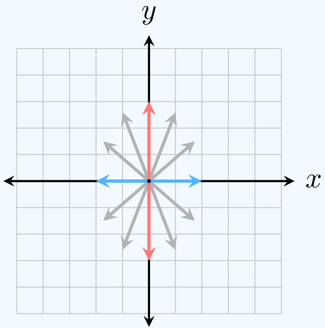
The transformation represented by the matrix

$$A = \begin{pmatrix} 1.75 & 0 \\ 0 & 0.5 \end{pmatrix}$$

scales each vector by 1.75 in the x-direction and by 0.5 in the y-direction. After aplication of the transformation, any vector on the x-axis remains on the x-axis (and is scaled by 1.75), and any vector on the y-axis remains on the y-axis (and is scaled by 0.5).

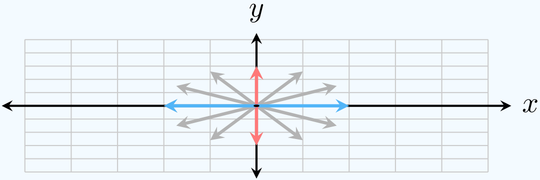
Definition

Example



Definition

Example



Definition

In matrix form, the general eigenvalue equation looks as follows:

$$A\vec{v} = \lambda\vec{v},$$

where $\lambda \in \mathbb{R}$ is the scalar by which $\vec{v}$ is streched after the application of A.

We call λ the **eigenvalue** corresponding to the eigenvector $\vec{v}$.

Definition

Example

In the previous example, the vectors lying on the x-axis have the corresponding eigenvalue $\lambda_1 = 1.75$, while the vectors lying on the y-axis have the corresponding eigenvalue $\lambda_2 = 0.5$.

Some Properties of Eigenvectors and Eigenvalues

Due to linearity, any scale of an eigenvector of a transformation T is also an eigenvector of the transformation, with the same eigenvalue.

Proof

Let A be a matrix with an eigenvector $\vec{v}$. Then for any scale $\alpha\vec{v}$ ($\alpha \in \mathbb{R}$):

$$A\left(\alpha\vec{v}\right) = \alpha A\vec{v} = \alpha\left(\lambda\vec{v}\right) = \lambda\left(\alpha\vec{v}\right).$$

Some Properties of Eigenvectors and Eigenvalues

All the linearly independent eigenvectors of a transformation $T : \mathbb{R}^n \to \mathbb{R}^n$ form a subspace of $\mathbb{R}^n$.

Some Properties of Eigenvectors and Eigenvalues

Linearly independent eigenvectors can have the same eigenvalues.

Example

The matrix

$$A = \begin{pmatrix} 1 & 0 & 1 \\ -2 & 3 & 1 \\ -2 & 0 & 4 \end{pmatrix}$$

has three linearly independent eigenvectors:

$$\vec{v}_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix},\ \vec{v}_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix},\ \vec{v}_3 = \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix},$$

with respective eigenvalues $\lambda_1 = 2,\ \lambda_2 = \lambda_3 = 3.$

Some Properties of Eigenvectors and Eigenvalues

Definition

The number of linearly independent vectors with the same eigenvalue is called the **geometric multiplicity** of the eigenvalue

Example

In the previous matrix, the eigenvalue $\lambda = 2$ has a geometric multiplicity of 1, and the eigenvalue $\lambda = 3$ has a geometric multiplicity of 2.

Finding the Eigenvectors of a Matrix

We can rearrange the eigenvalue equation

$$A\vec{v} = \lambda\vec{v}$$

to the form

$$A\vec{v} - \lambda\vec{v} = \vec{0},$$

and group $\vec{v}$ together, yielding

$$(A - \lambda I)\,\vec{v} = \vec{0}.$$

We get that $\vec{v}$ is the null space of the matrix $A - \lambda I$.

Finding the Eigenvectors of a Matrix

Since we assume that $\vec{v} \neq \vec{0}$ (otherwise the eigenvalue equation is somewhat pointless), this means that $|A - \lambda I| = 0$, since the null space of $A - \lambda I$ has more than just the zero vector.

The expression $P(\lambda) = |A - \lambda I|$ is actually a polynomial equation, due to way determinants are calculated. We therefore call $P(\lambda)$ the **characteristic polynomial** of the matrix A.

Solving for $P(\lambda) = 0$ yields all the eigenvalues of A.

Finding the Eigenvectors of a Matrix

Example

The characteristic polynomial of the matrix

$$A = \begin{pmatrix} 1 & 0 \\ -1 & 3 \end{pmatrix}$$

is

$$P(\lambda) = \begin{vmatrix} 1-\lambda & 0 \\ -1 & 3-\lambda \end{vmatrix} = (1-\lambda)(3-\lambda) - \cancel{0 \cdot (-1)}.$$

Thus, the solutions for $P(\lambda) = 0$ are $\lambda_1 = 1,\ \lambda_2 = 3$.

Example

We therefore know that the matrix A has some eigenvector with eigenvalue $\lambda = 1$. Let's find it: we want to multiply A by a generic vector, and equate the solution to the generic vector (meaning that it has an eigenvalue $\lambda = 1$).

$$\begin{pmatrix} 1 & 0 \\ -1 & 3 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x \\ -x + 3y \end{pmatrix} = 1 \cdot \begin{pmatrix} x \\ y \end{pmatrix},$$

this will happen when $x = 1, y = 0.5$, i.e. the vector $\vec{v}_1 = \begin{pmatrix} 1 \\ 0.5 \end{pmatrix}$ is an eigenvector of A. Let's verify this by applying A to $\vec{v}_1$.

Example

This yields

$$\begin{pmatrix} 1 & 0 \\ -1 & 3 \end{pmatrix} \begin{pmatrix} 1 \\ 0.5 \end{pmatrix} = \begin{pmatrix} 1+0 \\ -1+1.5 \end{pmatrix} = \begin{pmatrix} 1 \\ 0.5 \end{pmatrix},$$

i.e. $\vec{v}_1$ is indeed an eigenvector of A with $\lambda = 1$.

Example

Now for $\lambda = 3$:

$$\begin{pmatrix} 1 & 0 \\ -1 & 3 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = 3 \cdot \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x \\ -x + 3y \end{pmatrix}.$$

The solution in this case is $x = 0, \; y = 1$, i.e. the vector $\vec{v}_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$. Verifying:

$$\begin{pmatrix} 1 & 0 \\ -1 & 3 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \cdot 0 + 0 \cdot 1 \\ -1 \cdot 0 + 3 \cdot 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 3 \end{pmatrix} = 3 \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

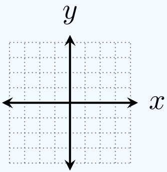
This means that $\vec{v}_2$ is indeed an eigenvector of A with $\lambda_2 = 3$.

Diagonalizing Matrices

Some matrices can represent complicated looking transformations but actually perform a simple scaling if we change our perspective.

Example

The matrix $A = \begin{pmatrix} 1.25 & 0.75 \\ 0.75 & 1.25 \end{pmatrix}$ performs the following transformation:



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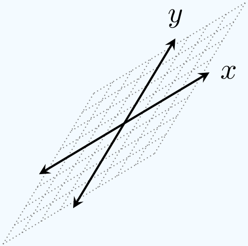
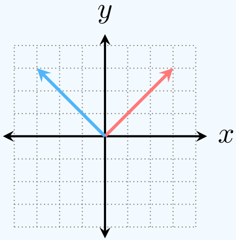


Example

We can rotate space such that its eigenvectors, $\vec{v}_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix} (\lambda_1 = 2)$ and $\vec{v}_2 = \begin{pmatrix} -1 \\ 1 \end{pmatrix} (\lambda_2) = 0.5$, are aligned with the axes:



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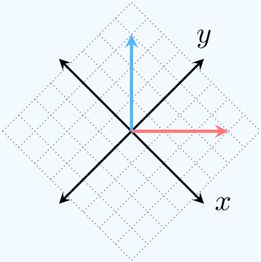
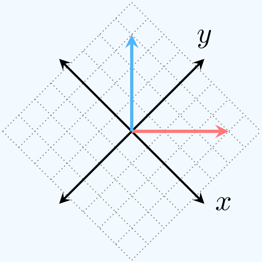


Example

In this perspective, applying A is simply a scaling by 2 in the x-direction and by 0.5 in the y-direction:



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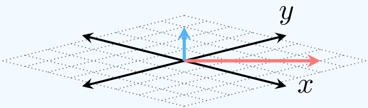

Diagonalizing Matrices

Example

This unisotropic scaling is expressed as a diagonal matrix:

$$D = \begin{pmatrix} 2 & 0 \\ 0 & 0.5 \end{pmatrix}.$$

To bring the diagonal matrix D "back" to be the original matrix A, we need to multiply it from both sides:

$$A = \begin{pmatrix} 1.25 & 0.75 \\ 0.75 & 1.25 \end{pmatrix} = \underbrace{\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}}_{P = \text{Eigenvectors of } A} \overbrace{\begin{pmatrix} 2 & 0 \\ 0 & 0.5 \end{pmatrix}}^{D,\ \text{Eigenvalues of } A} \underbrace{\begin{pmatrix} 0.5 & 0.5 \\ -0.5 & 0.5 \end{pmatrix}}_{P^{-1}}.$$

Diagonalizing Matrices

A matrix A that can be brought to a diagonal form is called a **diagonalizable matrix**. It can be **decomposed** as following:

$$A = P D P^{-1}$$

- P: Eigenvectors of A
- D: Eigenvalues of A
- P^{-1}: Inverse of P

A matrix which can't be diagonalized is called a **defective matrix**.