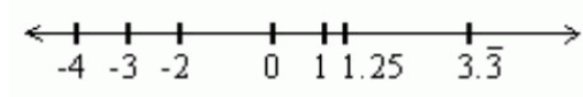


6. Numbers

We do not give an axiomatic definition of the *real numbers* here.

Intuitive meaning: Each point on the (infinite) line of numbers corresponds to a real number, i.e., an element of \mathbb{R} .

The line of numbers:



Important subsets of \mathbb{R} :

\mathbb{N} the set of all natural numbers

(positive integers), does not contain the 0

$\mathbb{N}_0 := \mathbb{N} \cup \{0\}$ the set of all non-negative integers

\mathbb{Z} the set of all integers $\{ \dots -2; -1; 0; 1; 2; \dots \}$

\mathbb{Q} the set of all rational numbers (representable as fractions of integers p/q , where $q \neq 0$)

We have:

$$\mathbb{N} \subseteq \mathbb{Z} \subseteq \mathbb{Q} \subseteq \mathbb{R}.$$

Remark:

Every rational number can be represented as decimal number with its expansion after the decimal dot either coming to an end or becoming periodic.

Examples:

$$1/4 = 0.25$$

$$1/7 = 0.\overline{142857} \quad (\text{periodic})$$

$$1/6 = 0.1\overline{6} \quad (\text{ultimately periodic})$$

Example for a transformation in the other direction:

$$0,\overline{62} = 0,62 \cdot \left(1 + \frac{1}{100} + \frac{1}{10000} + \frac{1}{1000000} + \dots\right) = 0,62 \cdot \frac{100}{99} = \frac{62}{99}$$

(note the different notations:

decimal dot in anglosaxon countries, comma in Germany)

Irrational numbers are real numbers that are not rational, i.e., cannot be expressed as a fraction of integers.

Their decimal expansion becomes never periodic.

Examples:

$$\sqrt{2} = 1,41421\ 35623\ 73095\ 04880\ 16887\ 24209\ 69807\ 85696\ 71875\ 37694\ \dots$$

$$\pi = 3,14159\ 26535\ 89793\ 23846\ 26433\ 83279\ 50288\ 41971\ 69399\ 37510\ \dots$$

$$e = 2,71828\ 18284\ 59045\ 23536\ 02874\ 71352\ 66249\ 77572\ 47093\ 69995\ \dots$$

Arithmetic operations on \mathbb{R} :

Addition

Operation symbol: +

$a + b$ exists for every $a, b \in \mathbb{R}$.

+ can be seen as a function with two arguments:

$a + b$ is in prefix notation $+(a, b)$.

Rules for adding numbers:

$$a + b = b + a \quad (\text{commutativity})$$

$$(a + b) + c = a + (b + c) \quad (\text{associativity})$$

$$a + 0 = a \quad (0 \text{ is the } \textit{neutral element} \text{ of addition})$$

For every a , there is a number $-a$ such that

$$a + (-a) = 0$$

We have always: $-(-a) = a$.

Subtraction can be derived from addition:

$$a - b = a + (-b).$$

Multiplication

Operation symbol: \cdot (often omitted!) (sometimes also $*$ instead of \cdot).

$a \cdot b$ exists for every $a, b \in \mathbb{R}$.

Rules for multiplication:

$$a \cdot b = b \cdot a$$

$$(a \cdot b) \cdot c = a \cdot (b \cdot c)$$

$$a \cdot 1 = a \quad (1 \text{ is the } \textit{neutral element} \text{ of multiplication})$$

Rule combining addition and multiplication:

$$a \cdot (b + c) = a \cdot b + a \cdot c \quad (\text{distributivity})$$

Note: By convention, \cdot binds stronger than $+$

For every $a \neq 0$, there is a number $1/a$ such that

$$a \cdot 1/a = 1.$$

We have always: $1/(1/a) = a$.

Other notations for $1/a$: $\frac{1}{a}$, a^{-1} .

$1/a$ is called the *inverse* of a .

Division can be derived from multiplication:

$$a : b = a \cdot 1/b$$

Another notation for $a : b$ is $\frac{a}{b}$.

$a : b$ is not defined for $b = 0$.

The power of a number

A power with a positive integer exponent is defined as an iterated multiplication:

Example: $4^3 = 4 \cdot 4 \cdot 4$.

4 is called the basis, 3 the exponent.

By definition, $a^0 = 1$ for all $a \neq 0$.

For $n > 0$, we define as the power with negative exponent $-n$: $a^{-n} = 1/(a^n)$ ($= (a^n)^{-1}$).

Example: $4^{-3} = (4^3)^{-1} = \frac{1}{4^3} = \frac{1}{64}$

The root of a number

For every positive real number a and every positive integer n there exists a positive real number x which fulfills the equation $x^n = a$.

This (unique) x is called the n -th root of a .

Two notations for x :

$$\sqrt[n]{a} = a^{\frac{1}{n}} \quad \forall a \in \mathbb{R}$$

For *odd* integers n and negative a we can extend this definition by $a^{1/n} = -(-a)^{1/n}$.

For *even* n , the n -th root of a negative number is not defined in \mathbb{R} .

To overcome this restriction, it is possible to extend the set of real numbers \mathbb{R} :

The so-called imaginary unit $i = \sqrt{-1}$ is defined which fulfills

$$i \cdot i = -1.$$

\mathbb{R} is extended to the set \mathbb{C} of *complex numbers*. Each complex number has the form $a + b \cdot i$ with $a, b \in \mathbb{R}$.

It is possible to calculate with complex numbers in the same way as with real numbers.

Visualization as points in the plane (with real-valued coordinates a, b).

Back to the real numbers:

The operation " n -th root of..." does invert the power operation.

Attention:

We have (by definition) $(\sqrt{x})^2 = x$,

but: $\sqrt{x^2} = |x|$!

Here, $|x|$ denotes the *absolute value* of x :

$|x| = x$ if $x \geq 0$ and $|x| = -x$ otherwise.

$|a - b|$: the *distance* between a and b .

In the context of square roots, the solution formula for quadratic equations ("*pq formula*") is often a useful tool:

For the equation $x^2 + px + q = 0$, the solutions (if they exist) are:

$$x_{1,2} = \frac{-p}{2} \pm \sqrt{\frac{p^2}{4} - q}$$

Condition for the existence of the solution(s):

$$\frac{p^2}{4} - q \geq 0$$

For control purposes, Vieta's theorem can be useful:

The two solutions fulfill $x_1 + x_2 = -p$
and $x_1 \cdot x_2 = q$.

The power of real numbers with rational exponent:

The power $a^{k/n}$ is defined as

$$a^{\frac{k}{n}} = \sqrt[n]{a^k}$$

(By using limits of series of rational numbers – for the introduction of limits see later – , the definition of a power can also be extended to irrational exponents.)

Rules for powers:

$$a^r \cdot a^s = a^{r+s}$$

$$a^r : a^s = a^{r-s}$$

$$(a^r)^s = a^{rs}$$

$$a^r \cdot b^r = (a \cdot b)^r$$

Because the power operation a^n is not commutative, there are two different reverse operations: You can search for a basis or you can search for an exponent. The first case leads to the root, the second case to the *logarithm*.

Definition:

Let $a, b > 0$ be real numbers. The (unique) solution of $b^x = a$ is $x = \log_b a$ (logarithm of a to the base b).

Often the so-called *natural logarithm* is used, which uses the Euler number $e = 2.718281828\dots$ as its base: $\ln a = \log_e a$.

Other frequent cases: binary logarithm (base 2); decimal logarithm (base 10).

In general, we have: $\log_b a = \ln a / \ln b$.

Rules for logarithms (hold for arbitrary base):

$$\log(x \cdot y) = \log x + \log y$$

$$\log(x / y) = \log x - \log y$$

$$\log(x^y) = y \cdot \log x$$

$$\log(\sqrt[n]{x}) = \frac{1}{n} \cdot \log x$$

The order relation on IR

Every two real numbers a, b can be ordered:

Either $a < b$, or $a = b$, or $a > b$.

$a \leq b$ means $a < b$ or $a = b$.

We have:

$a < b \Rightarrow a + c < b + c$ (analogously for \leq),

for $c > 0$: $a < b \Rightarrow a \cdot c < b \cdot c$

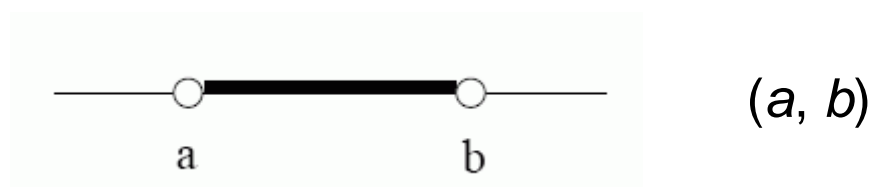
but for $c < 0$: $a < b \Rightarrow a \cdot c > b \cdot c$

Bounded intervals

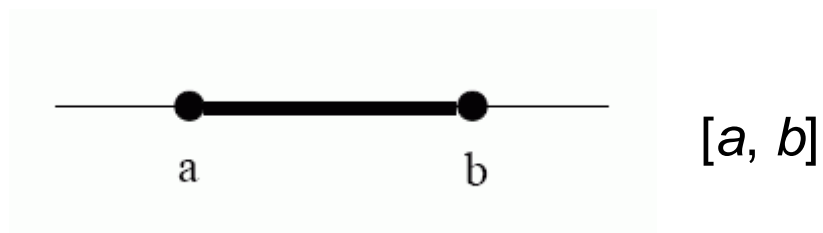
An *open, bounded interval* (a, b) is the set of all real numbers x which are *properly between* a and b , i.e., which fulfill $a < x < b$.

Attention! The same notation as for ordered pairs is used, but the meaning is different.

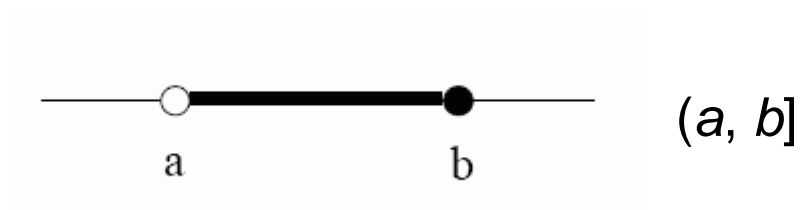
If $a < b$, (a, b) is an infinite set.



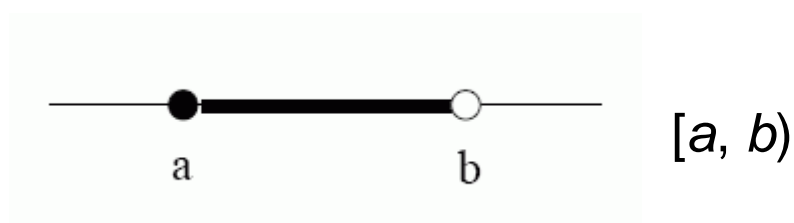
In a *closed interval* $[a, b]$, the end points are included: $[a, b] = \{ x \in \mathbb{R} \mid a \leq x \leq b \}$.



An interval closed on the right-hand side:



An interval closed on the left-hand side:



Unbounded intervals

$(a, +\infty) = \{ x \in \mathbb{R} \mid a < x \}$.



$$(a, +\infty)$$



$$[a, +\infty)$$

analogously for intervals unbounded to the left:



$$(-\infty, a)$$



$$(-\infty, a]$$

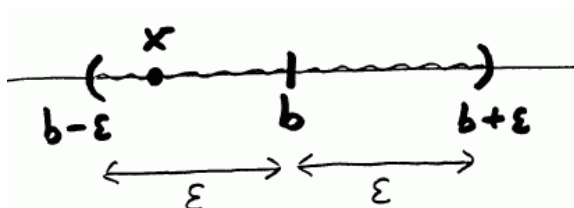
The *neighbourhood* of a number

Let $\varepsilon > 0$ be a positive real number.

The interval $(b - \varepsilon, b + \varepsilon)$ is called the ε -neighbourhood of the number b .

We have $(b - \varepsilon, b + \varepsilon) = \{x \in \mathbb{R} \mid |x - b| < \varepsilon\}$.

That means: The neighbourhood contains all numbers for which the *distance* to b is smaller than the given threshold ε .



Bounds

An *upper bound* of a set M of real numbers is a number r with $r > x$ for all $x \in M$.

Analogously: *lower bound* (exchange $>$ by $<$).

A set of numbers is called *bounded* if there exists an upper bound and a lower bound for it.

If a set has an upper bound, it has infinitely many upper bounds. We are interested in the smallest one:

The smallest upper bound of a set $M \subseteq \mathbb{R}$ is called the *supremum* of M , denoted $\sup M$.

Analogously:

The largest lower bound of a set $M \subseteq \mathbb{R}$ is called the *infimum* of M , denoted $\inf M$.

Examples:

$$\inf \{1; 2; 3; 4\} = 1, \quad \sup \{1; 2; 3; 4\} = 4,$$

$$\inf \left\{ \frac{1}{n} \mid n \in \mathbb{N} \right\} = 0$$

Number systems

Question: How to represent numbers?

We concentrate on positive integers here.

Decimal number system: base 10; each digit represents a multiple of an exponent of 10. Digits 0..9.

Example: $123.456_{10} = 1 * 10^2 + 2 * 10^1 + 3 * 10^0 + 4 * 10^{-1} + 5 * 10^{-2} + 6 * 10^{-3}$.

Binary number system: base 2. Only two digits: 0 and 1.

Example: $1101.01_2 = 1 * 2^3 + 1 * 2^2 + 0 * 2^1 + 1 * 2^0 + 0 * 2^{-1} + 1 * 2^{-2} = 13.25_{10}$.

Hexadecimal system (better but unhistorical name: sedecimal number system): Base 16, digits 0..9,A..F. One digit for four bits. Examples: $A2.8_{16} = 162.5_{10}$, $FF_{16} = 255_{10}$.

The additional digits in the hexadecimal system:

$A = 10$, $B = 11$, $C = 12$, $D = 13$, $E = 14$, $F = 15$.

Transformation from one number system to the other:

- Special case (easy): from binary to hexadecimal
Every 4 binary digits correspond directly to a hexadecimal digit

Example: $\underline{0000} \underline{0010} \underline{1100} \underline{0110}$
→ 0 2 C 6

- from arbitrary system to decimal:
Horner scheme

Input: $z_{n-1} z_{n-2} \dots z_0$ to base b

start with $h_{n-1} = z_{n-1}$

calculate for $k = n-1, n-2, \dots, 1$:

$$h_{k-1} = h_k * b + z_{k-1}$$

Output: $z = h_0$

Example:

Input: binary number 1010 ($n = 4, b = 2$)

Start: $h_{n-1} = h_3 = z_3 = 1$

$k = n-1 = 3$: $h_2 = h_3 * 2 + z_2 = 1*2 + 0 = 2$

$k = 2$: $h_1 = h_2 * 2 + z_1 = 2*2 + 1 = 5$

$k = 1$: $h_0 = h_1 * 2 + z_0 = 2*5 + 0 = \mathbf{10} = z$

- from decimal to arbitrary:
Inverse Horner scheme

start with $h_0 = z$ (= input)

calculate for $k = 1, 2, 3, \dots$:

$$z_{k-1} = h_{k-1} \bmod b,$$

$$h_k = h_{k-1} \operatorname{div} b$$

(mod: rest when dividing by b , div: integral part from dividing by b)

Output: $z_{n-1} z_{n-2} \dots z_0$ to base b

Example:

Input: decimal number 34, transform in ternary system ($b = 3$)

Start: $h_0 = 34$

$$k = 1: \quad z_0 = h_0 \bmod 3 = 34 \bmod 3 = 1, \\ \quad \quad \quad h_1 = h_0 \operatorname{div} 3 = 34 \operatorname{div} 3 = 11$$

$$k = 2: \quad z_1 = h_1 \bmod 3 = 11 \bmod 3 = 2, \\ \quad \quad \quad h_2 = h_1 \operatorname{div} 3 = 11 \operatorname{div} 3 = 3$$

$$k = 3: \quad z_2 = h_2 \bmod 3 = 3 \bmod 3 = 0, \\ \quad \quad \quad h_3 = h_2 \operatorname{div} 3 = 3 \operatorname{div} 3 = 1,$$

$$k = 4: \quad z_3 = h_3 \bmod 3 = 1 \bmod 3 = 1, \\ \quad \quad \quad h_4 = h_3 \operatorname{div} 3 = 1 \operatorname{div} 3 = 0 \text{ (Stop)}$$

$\Rightarrow z = 1021$

Remark:

Arbitrary real numbers can also be represented using an arbitrary integer $b > 1$ as base.

Digits after the dot are interpreted as coefficients of b^{-n} ($n = 1, 2, 3, \dots$).

Example:

$$0.111_2 \text{ (base } b=2) = 1/2 + 1/4 + 1/8 = 7/8 = 0.875_{10}$$

7. Vectors

We will work with elements from the set

$$\mathbb{R}^n = \mathbb{R} \times \mathbb{R} \times \mathbb{R} \times \dots \times \mathbb{R}$$

The elements are n -tuples of real numbers, we call them *vectors*.

To distinguish vector-valued variables from variables standing for single numbers, often an arrowed letter (\vec{a}) or printing in a different font is used.

Two ways to write down a vector:

row vector, e.g., (1; 5; -2)

column vector $\begin{bmatrix} 1 \\ 5 \\ -2 \end{bmatrix}$

To distinguish real numbers from vectors, we call them also *scalars*:

$$\vec{a} = \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{pmatrix} \in \mathbb{R}^n \quad \underline{\text{vector}}$$

(for $n = 2; 3$ geometrically:
representation by arrow ; “directed entity“)

$$m \in \mathbb{R} \quad \underline{\text{scalar}} \quad (\text{“undirected entity“})$$

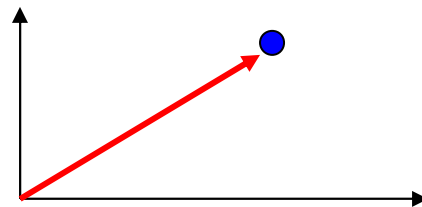
$\begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix}$... a_1, a_2, \dots are called *components* of the vector
(also: *coordinates*)

special cases:

$$\mathbb{R}^1 = \mathbb{R}$$

$\mathbb{R}^2 = \mathbb{R} \times \mathbb{R}$, can be represented as a plane:

each vector $\begin{bmatrix} x \\ y \end{bmatrix} \in \mathbb{R}^2$ corresponds to a point in the plane. Often a vector is represented as an arrow pointing from the origin to this point.

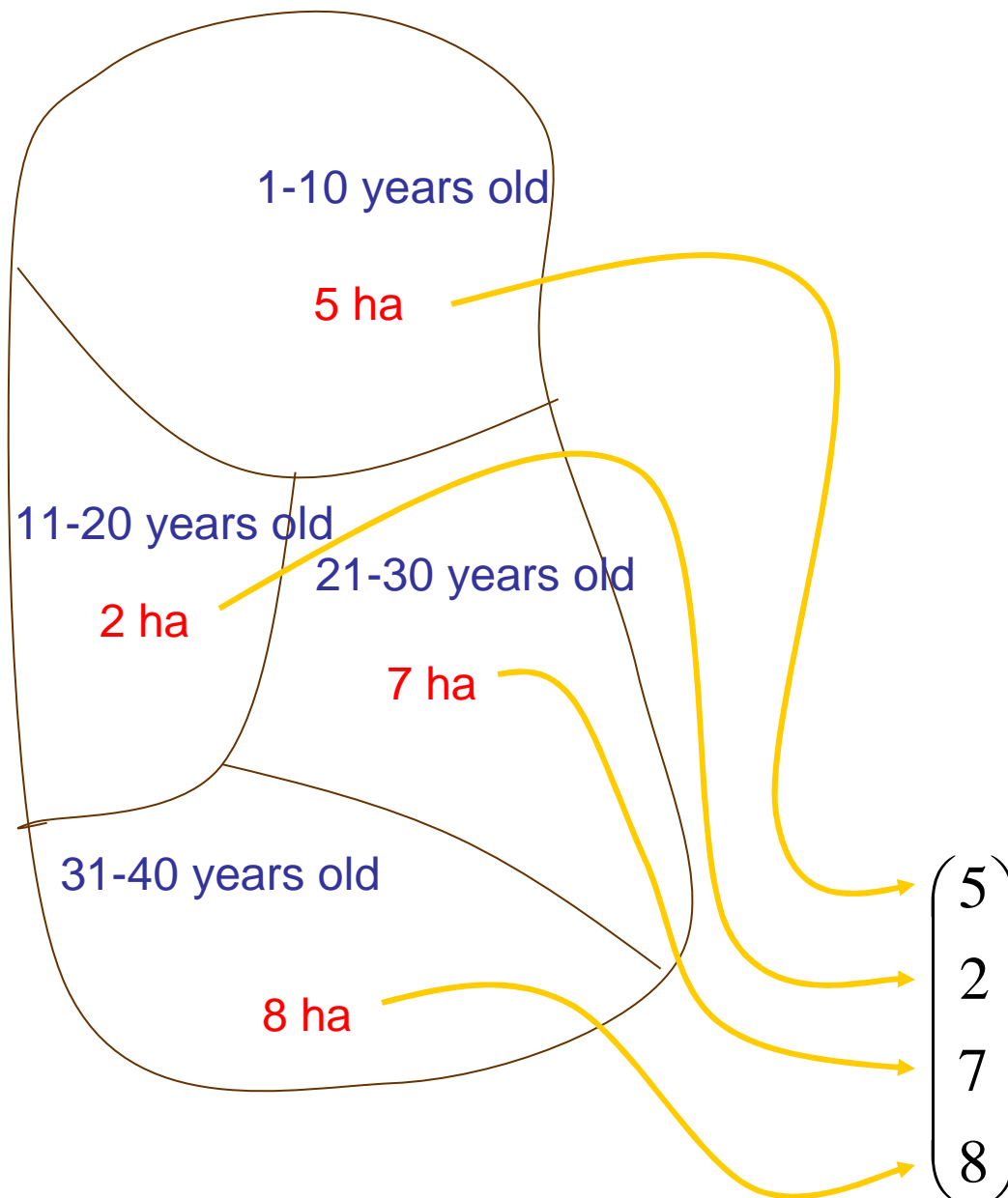


$\mathbb{R}^3 = \mathbb{R} \times \mathbb{R} \times \mathbb{R}$ 3-dimensional space.

\mathbb{R}^n is called an *n-dimensional vector space*.

Example of a vector in a higher-dimensional vector space \mathbb{R}^n ($n > 3$) :

The age-class vector of a population (e.g., of a forest stand)



Equality of vectors:

Two vectors are equal iff all their corresponding components are equal.

$$\begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{pmatrix} \Leftrightarrow a_1 = b_1 \wedge a_2 = b_2 \wedge \dots \wedge a_n = b_n$$

Addition of vectors:

Definition of the sum of two vectors in \mathbb{R}^n

$$\begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{pmatrix} + \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{pmatrix} = \begin{pmatrix} a_1 + b_1 \\ a_2 + b_2 \\ \vdots \\ a_n + b_n \end{pmatrix}$$

Properties of the addition of vectors:

$$\vec{a} + \vec{b} = \vec{b} + \vec{a} \quad \text{commutativity}$$

$$(\vec{a} + \vec{b}) + \vec{c} = \vec{a} + (\vec{b} + \vec{c}) \quad \text{associativity}$$

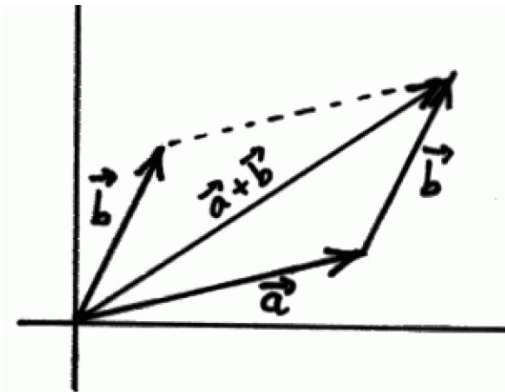
$$\vec{a} + \vec{0} = \vec{a}, \quad \text{neutral element } \vec{0}$$

where $\vec{0}$ is the zero vector:

$$\vec{0} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \in \mathbb{R}^n$$

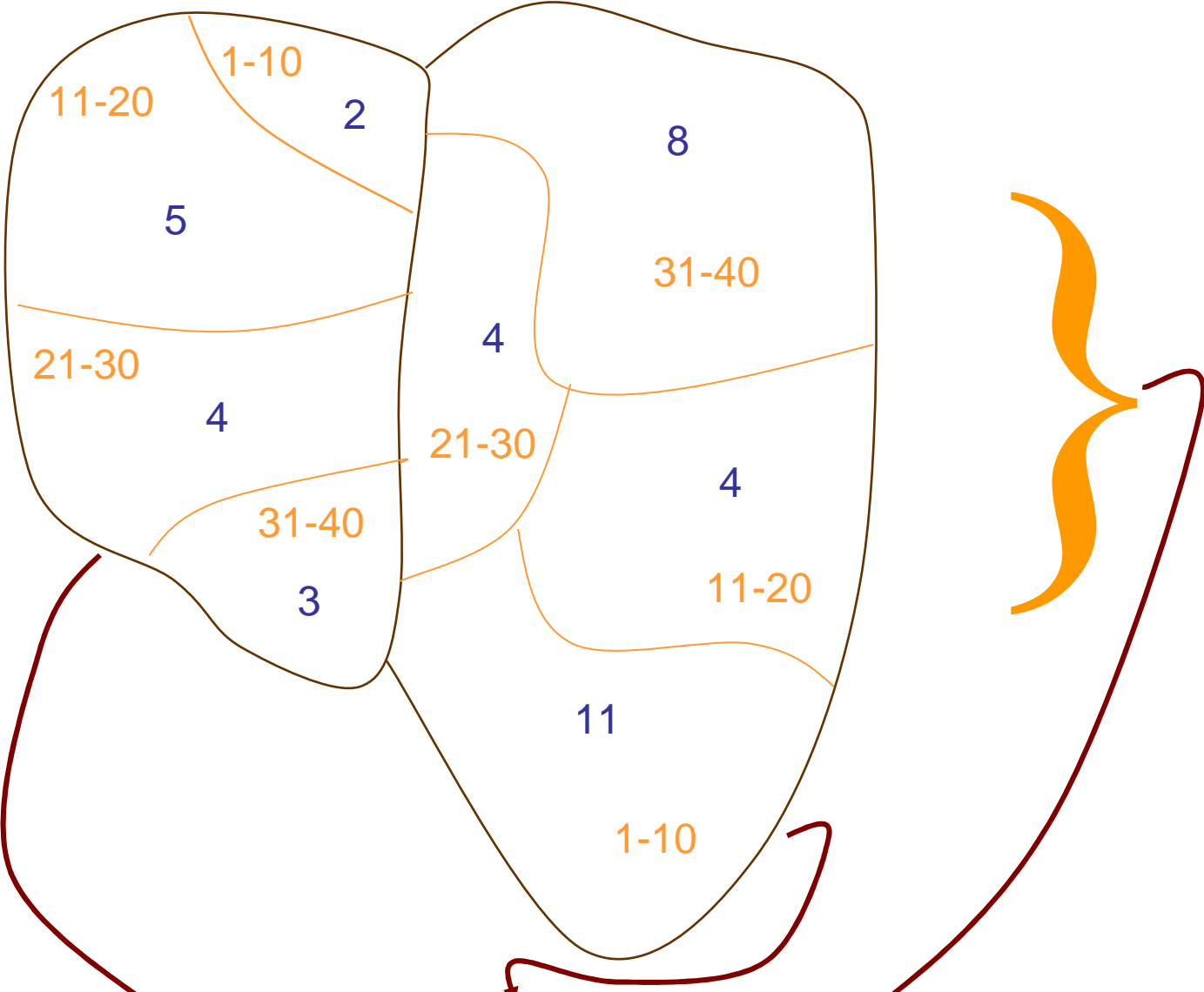
Geometrical interpretation of vector addition:

The arrows of both vectors are placed one after the other, and the origin is connected with the new end point.



(in physics: "parallelogram of forces")

The sum in the case of age-class vectors:
 aggregation of two forest stands into one.



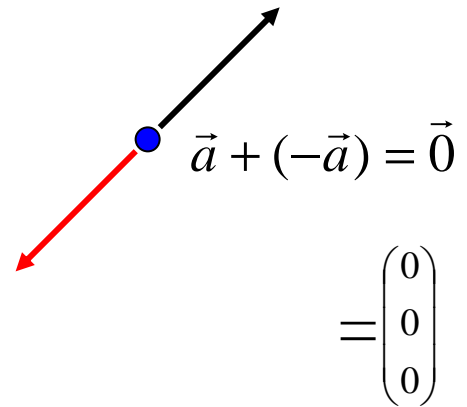
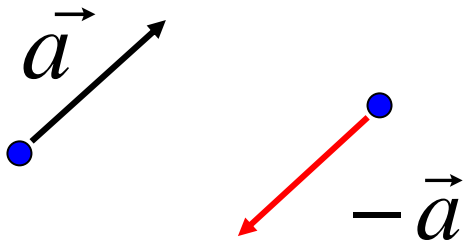
$$\begin{pmatrix} 2 \\ 5 \\ 4 \\ 3 \end{pmatrix} + \begin{pmatrix} 11 \\ 4 \\ 4 \\ 8 \end{pmatrix} = \begin{pmatrix} 13 \\ 9 \\ 8 \\ 11 \end{pmatrix}$$

age-class structure of the total area

For all vectors \vec{a} from \mathbb{R}^n , there exists exactly one vector $-\vec{a}$ which fulfills $\vec{a} + (-\vec{a}) = \vec{0}$.

↑
inverse (negative) element

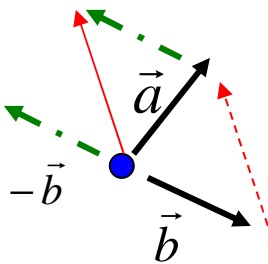
$$-\vec{a} = ?$$



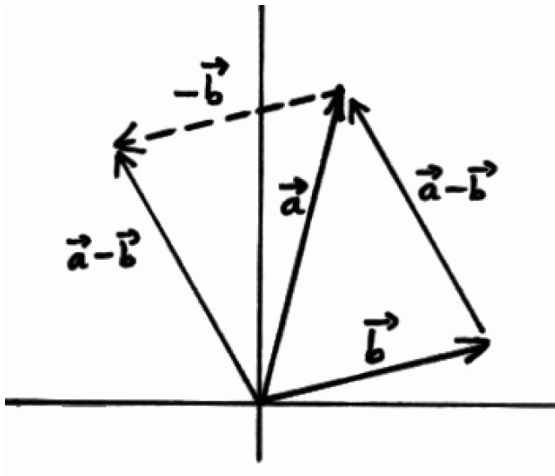
Difference of vectors:

$$\vec{a} - \vec{b}$$

$$= \vec{a} + (-\vec{b}) \quad (\text{as in the case of real numbers})$$



Geometrical interpretation of the difference of vectors:



$$\vec{a} - \vec{b} = \vec{a} + (-\vec{b})$$

↑

inversion of the direction

we get thus the "connecting vector" of the endpoints of both vectors.

Multiplication of a vector with a scalar (\neq „inner product“, \neq „vector product“ !)

$$m \in \mathbb{R}, \quad \vec{a} \in \mathbb{R}^n$$

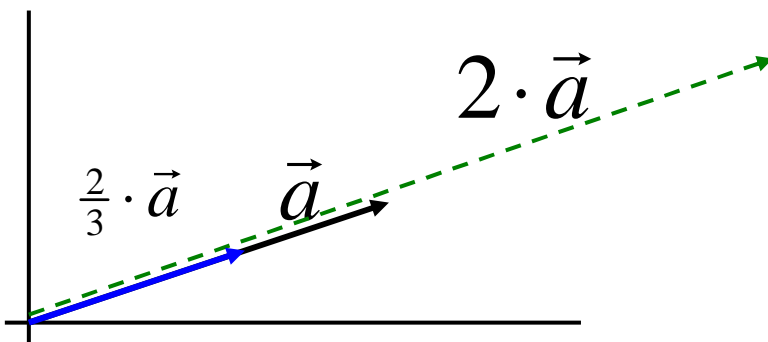
$$m \cdot \vec{a} = m \cdot \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{pmatrix} := \begin{pmatrix} m \cdot a_1 \\ m \cdot a_2 \\ \vdots \\ m \cdot a_n \end{pmatrix} \in \mathbb{R}^n$$

Example:

$$\frac{2}{3} \cdot \begin{pmatrix} 9 \\ -5 \\ 3 \end{pmatrix} = \begin{pmatrix} \frac{2}{3} \cdot 9 \\ \frac{2}{3} \cdot (-5) \\ \frac{2}{3} \cdot 3 \end{pmatrix} = \begin{pmatrix} 6 \\ -\frac{10}{3} \\ 2 \end{pmatrix}$$

geometrical meaning:

expansion, resp. compression of \vec{a} by the factor m



The direction is inverted, if the factor m is < 0 .

We have the following rules:

$$1 \cdot \vec{a} = \vec{a}$$

$$0 \cdot \vec{a} = \vec{0}$$

$$(-1) \cdot \vec{a} = -\vec{a}$$

$$m \cdot \vec{0} = \vec{0}$$

$$m \cdot \vec{a} = \vec{0} \Rightarrow m = 0 \vee \vec{a} = \vec{0}$$

$$\left. \begin{aligned} m \cdot (\vec{a} + \vec{b}) &= m \cdot \vec{a} + m \cdot \vec{b} \\ (k + m) \cdot \vec{a} &= k \cdot \vec{a} + m \cdot \vec{a} \end{aligned} \right\} \text{distributive laws}$$

In the following, terms of the form

$$m_1 \cdot \vec{a}_1 + m_2 \cdot \vec{a}_2 + \dots + m_k \cdot \vec{a}_k$$

$$\left(= \sum_{i=1}^k m_i \cdot \vec{a}_i \right), \quad m_i \in \mathbb{R}, \quad \vec{a}_i \in \mathbb{R}^n$$

are important. We speak of a **linear combination** of the vectors $\vec{a}_1, \dots, \vec{a}_k$; the m_i are called **coefficients**.

Example (in 3-dimensional space):

$$\vec{a}_1 = (1, -1, 0) \quad , \quad \vec{a}_2 = (2, 1, 1) \quad , \quad \vec{a}_3 = (-2, 0, 0) \quad ,$$

$$\vec{a}_4 = (0, -2, 2)$$

(here written as row vectors for convenience)

The vector

$$\vec{b} = 3\vec{a}_1 - 2\vec{a}_2 + 0\vec{a}_3 + 3\vec{a}_4$$

is a linear combination of these four vectors.

In column-vector notation, we calculate:

$$\vec{b} = 3 \cdot \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} - 2 \cdot \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix} + 0 \cdot \begin{bmatrix} -2 \\ 0 \\ 0 \end{bmatrix} + 3 \cdot \begin{bmatrix} 0 \\ -2 \\ 2 \end{bmatrix} = \begin{bmatrix} -1 \\ -11 \\ 4 \end{bmatrix}$$

The trivial linear combination

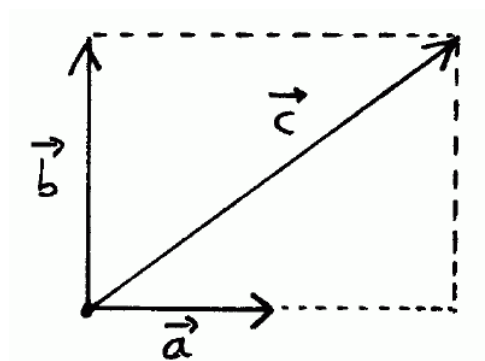
A linear combination is called *trivial* if all coefficients m_1, \dots, m_k are 0.

It is called nontrivial if at least one coefficient is *not* 0.

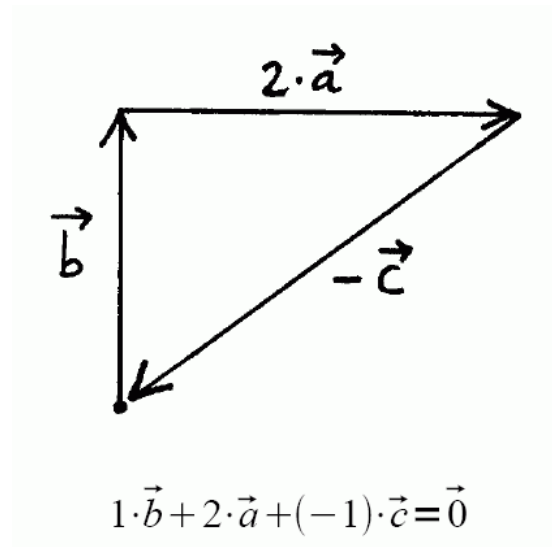
A trivial linear combination has the zero vector as its result.

Can the zero vector also be the result of a *nontrivial* linear combination?

An example: 3 vectors in a plane



We can indeed construct a "cycle" of multiples of these vectors which gives as its sum the zero vector:



This is a *nontrivial* linear combination giving the zero vector!

$0 \cdot \vec{b} + 0 \cdot \vec{a} + 0 \cdot \vec{c} = \vec{0}$ would be trivial.

We say: $\vec{a}, \vec{b}, \vec{c}$ are *linearly dependent*.

Definition:

Linear dependence / independence of vectors

Given are $k \in \mathbb{N}$ and the vectors

$$\vec{a}_1, \vec{a}_2, \dots, \vec{a}_k \in \mathbb{R}^n .$$

These vectors are called *linearly dependent*, if there exist real numbers m_1, \dots, m_k , which are *not all equal to zero*, such that

$$\sum_{i=1}^k m_i \vec{a}_i = \vec{0} .$$

If the latter equation holds only if all coefficients are 0, then the vectors are called *linearly independent*.

One can prove: Several vectors are linearly dependent if and only if *one of them can be represented as a linear combination of the others*.

Special cases:

\mathbb{R}^1 : only sets with one element, $\{ a \}$, with $a \neq 0$ are linearly independent.

\mathbb{R}^2 : $\{ \vec{a}_1, \vec{a}_2 \}$ is linearly dependent \Leftrightarrow both vectors are on a line through the origin.

\mathbb{R}^3 : $\{ \vec{a}_1, \vec{a}_2, \vec{a}_3 \}$ is linearly dependent \Leftrightarrow all three vectors are in a plane going through the origin of the coordinate system.

How to test a set of vectors for linear dependence

Example: Given are the three vectors $(1; 2; 3)$, $(0; -1; 0)$ and $(-1; 2; -2)$. Are they linearly dependent?

Approach: We have to assume $\sum_{i=1}^3 m_i \vec{a}_i = \vec{0}$.

Written with column vectors, this means:

$$\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} = m_1 \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} + m_2 \begin{bmatrix} 0 \\ -1 \\ 0 \end{bmatrix} + m_3 \begin{bmatrix} -1 \\ 2 \\ -2 \end{bmatrix}$$

For each component, we obtain an equation, giving together the following system of 3 linear equations:

$$0 = m_1 - m_3 \Rightarrow m_3 = m_1$$

$$0 = 2m_1 - m_2 + 2m_3$$

$$0 = 3m_1 - 2m_3 \Rightarrow 2m_3 = 3m_1$$

We can solve this step by step for the unknowns m_i . In this case, we obtain quickly $m_1 = m_2 = m_3 = 0$. So the system can only be fulfilled if all coefficients are zero, and the 3 vectors have been proven als *linearly independent*.

Examples for training:

Linearly dependent or independent? Decide yourself!

$$(a) \left\{ \begin{bmatrix} 3 \\ 3 \end{bmatrix}, \begin{bmatrix} -2 \\ -2 \end{bmatrix} \right\}$$

$$(b) \left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\}$$

$$(c) \left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right\}$$

$$(d) \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 5 \\ 4 \\ 0 \end{bmatrix} \right\}$$

$$(e) \left\{ \begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 3 \\ 3 \end{bmatrix} \right\}$$

Rank of a set of vectors

The number of elements of the *maximal* linearly independent subset of a given set of vectors is called the *rank* of the set of vectors.

The *basis* of a vector space

\mathbb{R}^n has infinitely many elements.

Is there a finite subset $\{\vec{a}_1, \dots, \vec{a}_k\}$, such that **all** vectors from \mathbb{R}^n can be represented **uniquely** as a linear combination of the \vec{a}_i ?

YES!

Such a set of vectors is called a *basis* of \mathbb{R}^n .


Most simple example of a basis:

$$\vec{e}_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \quad \vec{e}_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \quad \dots, \quad \vec{e}_n = \begin{pmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 1 \end{pmatrix},$$

the *standard basis* of \mathbb{R}^n .

There are infinitely many bases, which have, however, all the same number of elements (**namely, n**). This number is called the *dimension* of the vector space.

Example:

$$\left\{ \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 2 \end{pmatrix} \right\} \text{ has rank 2}$$


lin. indep. *lin. dependent*


If we remove $\begin{pmatrix} 0 \\ 0 \\ 2 \end{pmatrix}$, we obtain a linearly independent

vector system:

$$\left\{ \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right\}, \text{rank } 2.$$

If we add now, e.g., $\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$, we obtain a basis of \mathbb{R}^3 ,

i.e., a maximal linearly independent subset:

$$\left\{ \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \right\}, \text{rank } 3.$$


lin. independent

3 is the dimension of \mathbb{R}^3 .

If we add an arbitrary further element,

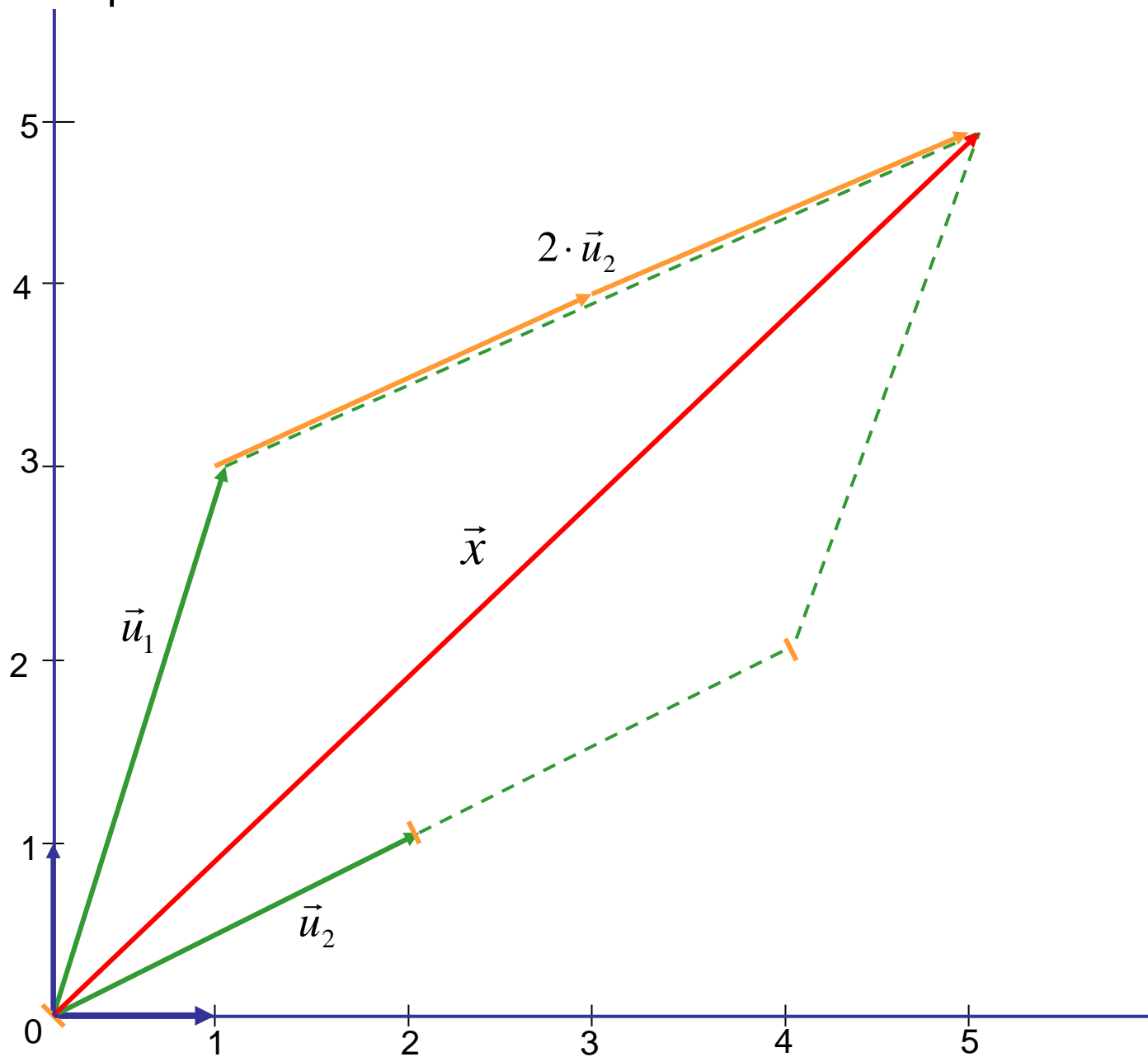
e.g., $\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$, the set becomes linearly dependent:

$$1 \cdot \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} + 0 \cdot \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} + (-1) \cdot \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + (-1) \cdot \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = \vec{0}.$$

The *coordinates* of a vector with respect to a given basis

When an arbitrary basis is given, every vector can be expressed uniquely as a linear combination of the elements of this basis (i.e., the coefficients are uniquely determined).

Example:



$$\vec{u}_1 = \begin{pmatrix} 1 \\ 3 \end{pmatrix}, \quad \vec{u}_2 = \begin{pmatrix} 2 \\ 1 \end{pmatrix}, \quad \{ \vec{u}_1, \vec{u}_2 \} \text{ basis of } \mathbb{R}^2$$

$$\vec{x} = \begin{pmatrix} 5 \\ 5 \end{pmatrix} = 1 \cdot \vec{u}_1 + 2 \cdot \vec{u}_2$$

(1; 2) are the coordinates of \vec{x}
w.r.t. $\{ \vec{u}_1, \vec{u}_2 \}$.

In the special case of the standard basis, we have always:

$$\begin{aligned} & a_1 \cdot \vec{e}_1 + a_2 \cdot \vec{e}_2 + \dots + a_n \cdot \vec{e}_n \\ &= a_1 \cdot \begin{pmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix} + a_2 \cdot \begin{pmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} + \dots + a_n \cdot \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix} \\ &= \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{pmatrix} \end{aligned}$$

The components a_1, \dots, a_n of a vector $\vec{a} \in \mathbb{R}^n$ are exactly the coordinates of \vec{a} with respect to the standard basis.