Computer Science and Mathematics

Part I:
Fundamental Mathematical Concepts

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1. Mathematical Logic

Propositions

- can be either true or false
- Examples: "Vienna is the capital of Austria", "Mary is pregnant", "3+4=8"
- can be combined by logical operators, e.g., "Today is Tuesday *and* the sun is shining".

Usual logical operators and their abbreviations:

$a \wedge b$	a and b	(**A**nd)
$a \vee b$	a or b	(latin: **v**el)
$\neg a$	not a	
$a \Rightarrow b$	a implies b (if a then b)	
$a \Leftrightarrow b$	a is equivalent to b (if and only if a then b; iff a then b)	

Quantifiers

$\forall x$	for all x holds ...
$\exists x$	there exists an x for which ...

Further symbols

$:=$	is equal by definition
$:\Leftrightarrow$	is equivalent by definition

2. Sets

A *set* is a collection of different objects, which are called the *elements* of the set.
The order in which the elements are listed does not matter.
A set can have a finite or an infinite number of elements. We speak of finite and infinite sets.

Examples:
The set of all human beings on earth (finite)
The set of all prime numbers (infinite)

Sets are usually designated by upper-case letters, their elements by lower-case letters.

$a \in M$	a is element of the set M
$a \notin M$	a is *not* element of the set M

Two notations for sets:

- Listing of all elements, delimited by commas (or semicolons) and put in braces:
  $A = \{ 1; 2; 3; 4; 5 \}$

- Usage of a variable symbol and specification of a proposition (containing the variable) which has to be fulfilled by the elements:
  $A = \{ x \mid x \text{ is a positive integer smaller than } 6 \}$
  (the vertical line is read: "... for which holds: ...")

  alternative notation for the last one:
  $A = \{ x \in \mathbb{N} \mid x < 6 \}$

($\mathbb{N}$ is the set of positive integer numbers, not including 0.)

Number of elements of a set *M* (also called *cardinality* of *M*): $|M|$

example: $|\{ x \in \mathbb{N} \mid x < 12 \wedge x \text{ is even} \}| = 5$

Propositions involving sets:
Example: $\exists n \in \mathbb{N} : n^2 = 2^n$
(true, because it is fulfilled for $n = 2$)

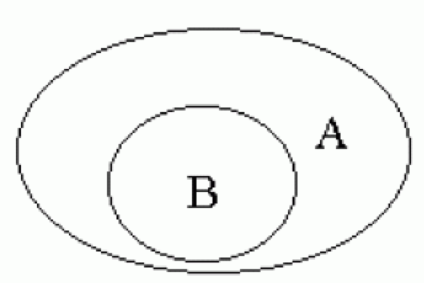
A special set: The *empty set*
Notation: $\emptyset$
For the empty set, we have $|\emptyset| = 0$.

Subsets and supersets

If A contains all elements of B (and possibly some more), B is called a *subset* of A (and A a *superset* of B).

Notation: $B \subseteq A$ (or, equivalently, $A \supseteq B$)

Visualization by a so-called *Venn diagram*:



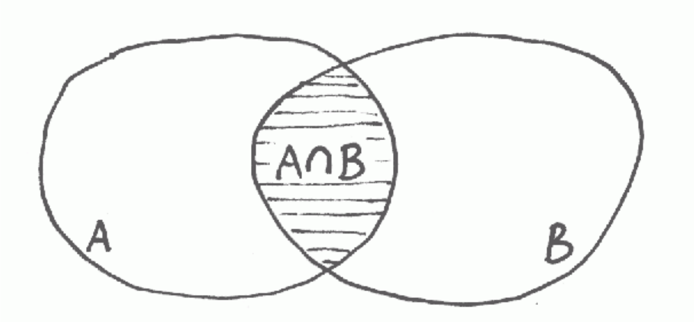
It holds: $A \subseteq B \;\wedge\; B \subseteq A \;\Leftrightarrow\; A = B$.

Intersection

The *intersection* of the sets A and B is the set of all elements which are elements of A *and* of B.
Operator symbol: $\cap$

$A \cap B = \{ x \mid x \in A \text{ and } x \in B \}$.

Example:
$\{ 1; 2; 3; 4 \} \cap \{ 2; 4; 6; 8 \} = \{ 2; 4 \}$.



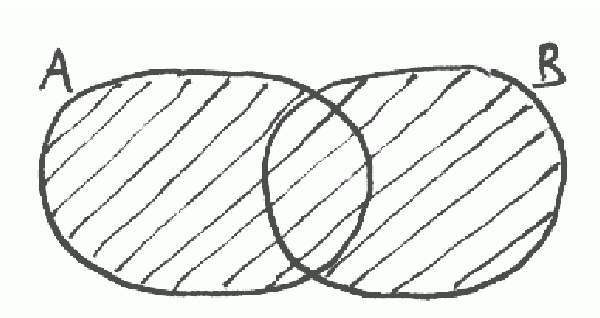
Two sets *A* and *B* are called disjoint if $A \cap B = \emptyset$.

Union

The *union* of the sets *A* and *B* is the set of all elements which are element of *A* *or* of *B*.
Operator symbol: $\cup$ ($\cup$ = **U**nion)

$A \cup B = \{ x \mid x \in A \text{ or } x \in B \}$.

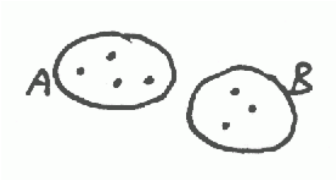
Example:
$\{ 1; 2; 3; 4 \} \cup \{ 2; 4; 6; 8 \} = \{ 1; 2; 3; 4; 6; 8 \}$.



What is the number of elements $| A \cup B |$?

If A and B are disjoint, we have:

$$|A \cup B| = |A| + |B|$$



Generalization:
If $A_1, A_2, \ldots, A_n$ are all pairwise disjoint, then
$|A_1 \cup A_2 \cup \ldots \cup A_n| = |A_1| + \ldots + |A_n|$.

Remarks:
(1) $(A \cup B) \cup C = A \cup (B \cup C)$ ("associativity"),

so we can omit the parentheses
(the same holds for + and for $\cap$).

(2) Short notations for iterated operations:

for n sets $A_1, A_2, \ldots, A_n$:

$$\bigcup_{i=1}^{n} A_i = A_1 \cup \ldots \cup A_n$$

for n numbers $x_1, x_2, \ldots, x_n$:

$$\sum_{i=1}^{n} x_i = x_1 + \ldots + x_n$$

$$\prod_{i=1}^{n} x_i = x_1 \cdot \ldots \cdot x_n$$

(3) The formula $|A \cup B| = |A| + |B|$ does not hold if A and B are not disjoint. In the general case, we have:

$$|A \cup B| = |A| + |B| - |A \cap B|.$$

Difference of sets

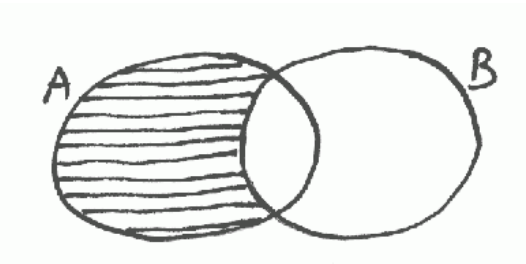
The *difference set* of the sets A and B is the set of all elements which are element of A *but not* of B. ("*A without B*")

Operator symbol: – (sometimes also used: \ ).

$A - B = \{ x \mid x \in A$ and $x \notin B \}$.

Example:

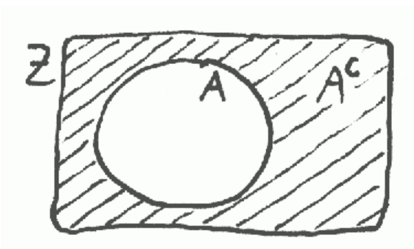
$\{ 1; 2; 3; 4 \} - \{ 2; 4; 6; 8 \} = \{ 1; 3 \}$.



Complement

If all considered sets are subsets of a given basic set Z, the difference $Z–A$ is often called the *complement* of A and is denoted A^C.

$A^C = \{ x \in Z \mid x \notin A \}$.



The power set

The set of all subsets of a given set S is called the *power set* of S and is denoted $\mathrm{P}(S)$.

$$\mathrm{P}(S) = \{ A \mid A \subseteq S \}$$

Example:

$S = \{ 1; 2; 3 \}$
$\mathrm{P}(S) = \{ \emptyset; \{1\}; \{2\}; \{3\}; \{1; 2\}; \{1; 3\}; \{2; 3\}; \{1; 2; 3\} \}$

For the number of its elements, we have always:

$$| \mathrm{P}(S) | = 2^{|S|}$$

Cartesian products of sets

The *cartesian product* of two sets A and B, denoted $A \times B$, is the set of all possible *ordered pairs* where the first component is an element of A and the second component an element of B.

$$A \times B = \{ (a, b) \mid a \in A \text{ and } b \in B \}$$

Remark: In an ordered pair, the order of the components is fixed. If $a \neq b$, then $(a, b) \neq (b, a)$.

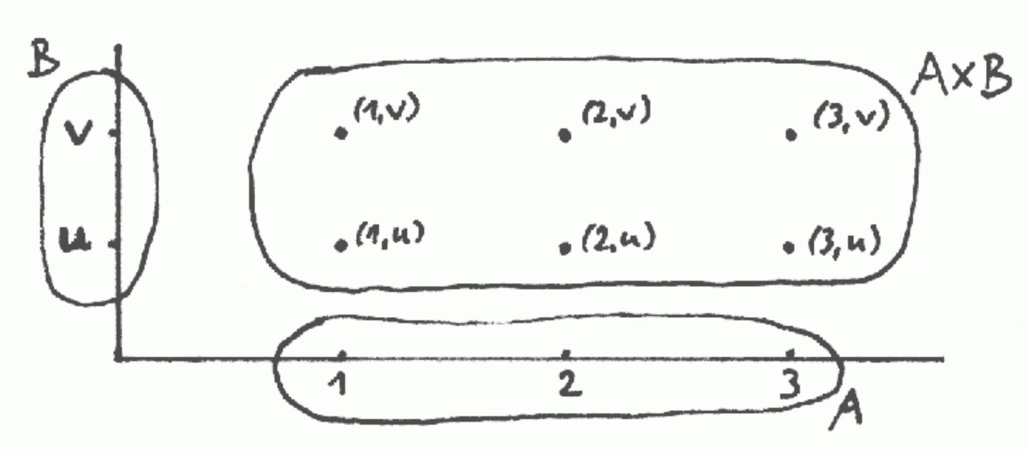
Example: $A = \{ 1; 2; 3 \}$, $B = \{ \text{u}, \text{v} \}$:

$$A \times B = \{ (1, \text{u}); (2, \text{u}); (3, \text{u}); (1, \text{v}); (2, \text{v}); (3, \text{v}) \}.$$

Attention: Usually it is $A \times B \neq B \times A$!

Number of elements: $|A \times B| = |A| \cdot |B|$.

Visualization of $A \times B$ in a coordinate system:



If A and B are subsets of the set IR of real numbers, we can use the well-known cartesian coordinate system.

Products of more than two sets

The elements of $(A \times B) \times C$ are "nested pairs" $((a, b), c)$; we identify them with the triples $\mathbf{(a, b, c)}$ and write $A \times B \times C$.
Analogously for quadruples, etc.

$$A_1 \times A_2 \times ... \times A_n = \{ (a_1, a_2, ..., a_n) \mid a_1 \in A_1 \wedge a_2 \in A_2 \wedge ... \wedge a_n \in A_n \}$$

If $A_1 = A_2 = ... = A_n$, we write:

$$A^n = \underbrace{A \times A \times ... \times A}_{(n\ times)}$$

= set of all n-tuples with components from A.

Example:

$B = \{ x, y \} \Rightarrow$

$B^3 = \{ (x, x, x); (x, x, y); (x, y, x); (x, y, y);$
$(y, x, x); (y, x, y); (y, y, x); (y, y, y) \}$

If the components are letters, the parentheses and commas are often omitted:

$B^3 = \{xxx; xxy; ...; yyy \}$ set of *words* of length 3

The set of arbitrary words (strings) over a set:

$$A^* = A^0 \cup A^1 \cup A^2 \cup A^3 \cup ...$$

with $A^0 := \{ \varepsilon \}$, where ε is the *empty word*.

Example: $\{ x, y \}^* = \{ \varepsilon; x; y; xx; xy; yx; yy; xxx; ... \}$

$A^+ = A^1 \cup A^2 \cup A^3 \cup ...$
does not contain the empty word.

The cartesian product in the description of datasets

Frequently, informations regarding a measurement are put together in an n-tuple. Example:

S = set of time values
T = set of temperature values
U = set of laboratory identifiers
V = set of measurement values

A measurement is then represented by a 4-tuple

$$(s, t, u, v) \in S \times T \times U \times V$$

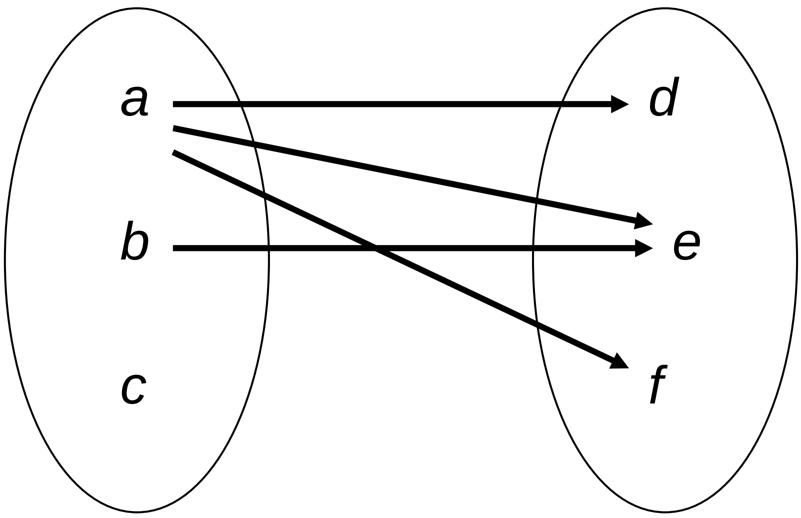
with s = time of measurement, t = current temperature, u = lab id, v = measured value.

3. Relations

A (binary) relation R between two sets A and B is a subset of $A \times B$.
That means, a relation is represented by a set R of ordered pairs (a, b) with $a \in A$ and $b \in B$.
If $(a, b) \in R$ we write also $a\ R\ b$ (infix notation).

Graphical representation (if A and B are finite):

If $(a, b) \in R$, connect a and b by an arrow



The converse relation R^{-1} of R:

$(b, a) \in R^{-1} \Leftrightarrow (a, b) \in R$

R^{-1} is a subset of $B \times A$.

In the graphical representation, switch the directions of all arrows to obtain the converse relation!

If $A = B$, we have a relation *in* a set A.

Example: $A = \mathbb{R}$ (set of real numbers), $R = <$ relation "smaller as". R consists of all number pairs (x, y) with $x < y$.

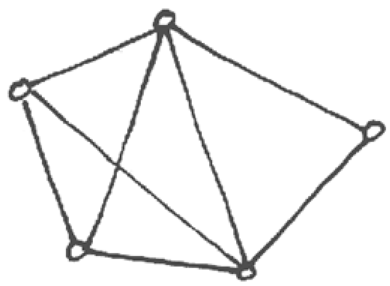
Generalization: n-ary relation:
any subset of $A_1 \times A_2 \times \ldots \times A_n$.

4. Graphs

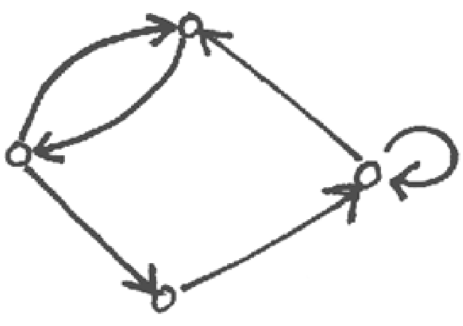
A graph consists of a set V of vertices and a set E of edges. Each edge connects two vertices.

Different variants of graphs differ in the way how the edges are defined and what edges are allowed:

• Undirected graphs:
The edges are (unordered) 2-element subsets of V.
Visualization by undirected arcs:

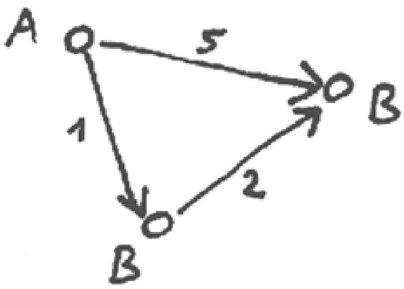


• Directed graphs:
The edges are ordered pairs, i.e., $E \subseteq V \times V$.
(E is a relation in V.)
Visualization by directed arcs. "Loops" are allowed, multiple arcs between the same vertices are not allowed:



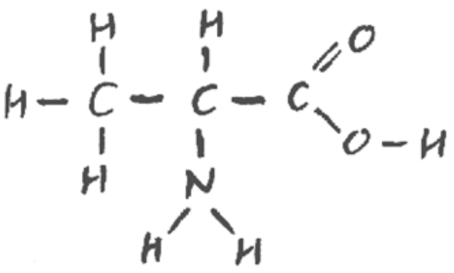
• Multigraphs:
Multiple directed edges are allowed.

• Labelled graphs:
Vertices and/or edges have labels from a set of vertex/edge labels (names, numbers...)



Examples:

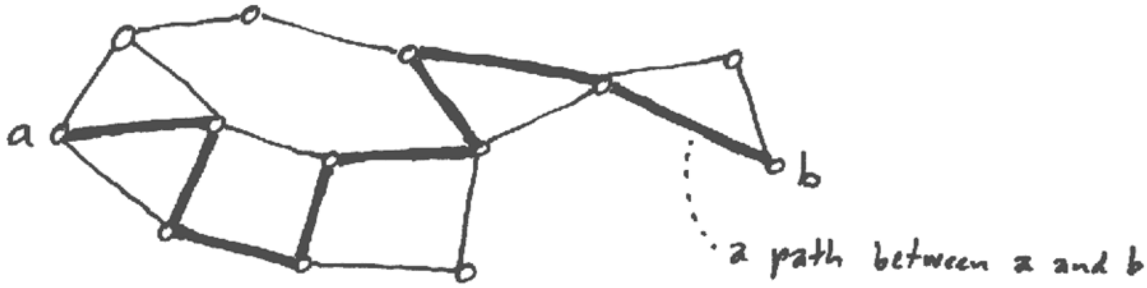
- transport networks
- metabolic networks
- food webs
- class diagrams in software engineering
- genealogical trees
- structural formulae in chemistry



(vertex-labelled multigraph)

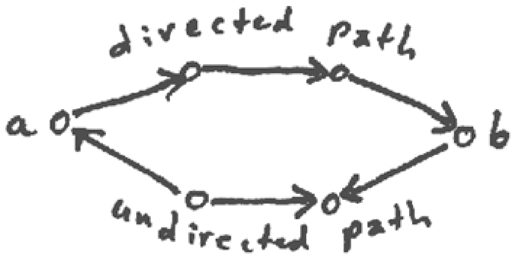
Paths in graphs:

A *path* is a sequence of edges where two consecutive edges have one vertex in common:

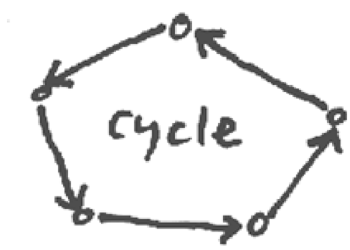


A path where start and end vertex coincide is called a *circle*.

In directed graphs, we distinguish between directed and undirected paths.

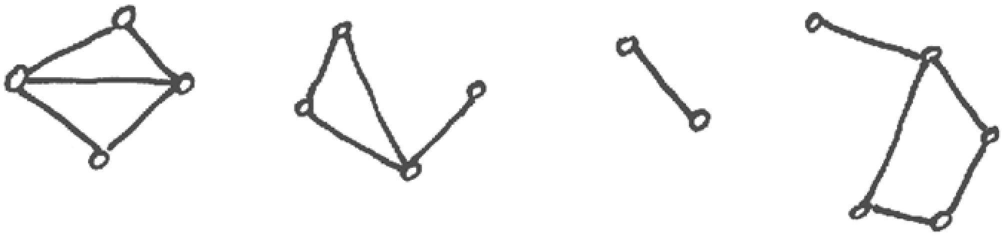


A directed circle is called a *cycle*.



Connectedness

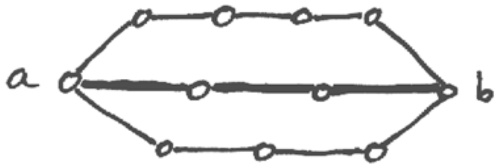
If for every pair of vertices (*a*, *b*) in a graph, there is a path between *a* and *b*, the graph is called *connected*.
Every unconnected graph can be decomposed in *connected components*.



A graph with 4 connected components

Graph-theoretical distance

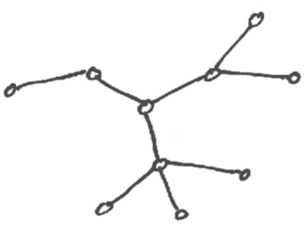
The *distance* between two vertices *a* and *b* in a graph is the length, i.e., the number of edges, of the shortest path between *a* and *b* – if such a path exists. Otherwise, the distance is undefined.



dist(*a*, *b*) = 3

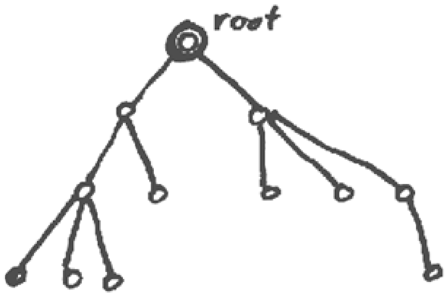
Trees

A *tree* is a graph without circles.

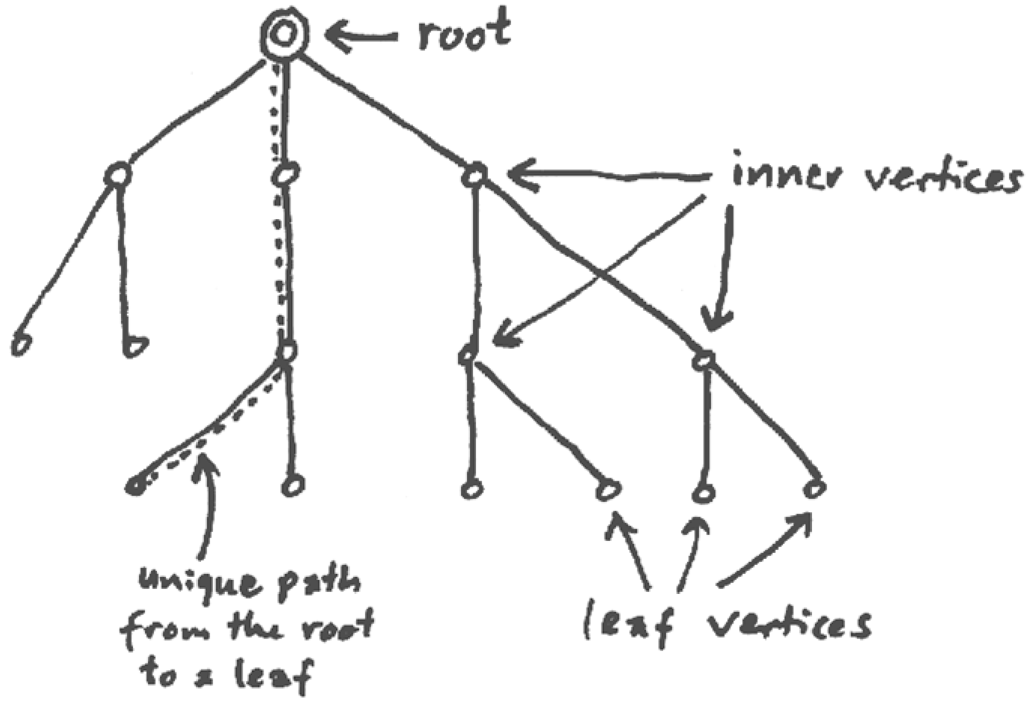
A tree: 

Example: phylogenetic trees, describing genetic kinship between species.

A *rooted tree* is a tree in which one vertex, the root, is distinguished.
The root is often drawn at the top:



Rooted trees are used to describe hierarchies, e.g., in biological systematics, in organisations or in nested directories of data.

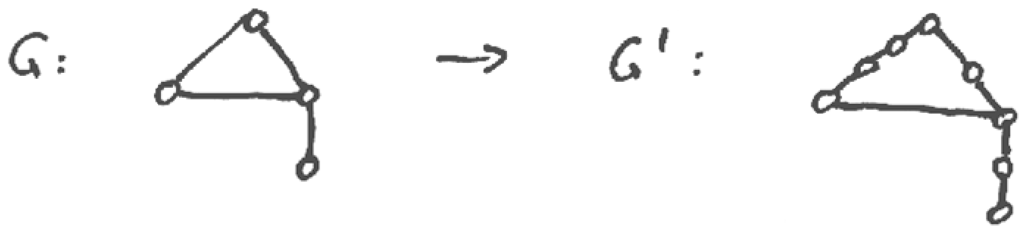


Degree

The number of edges to which a vertex belongs is called the *degree* of the vertex.
In directed graphs, we distinguish between *indegree* and *outdegree* of a vertex.

Subdivision

A *subdivision* G' of a graph G is obtained by inserting vertices of degree 2 in the edges of G.



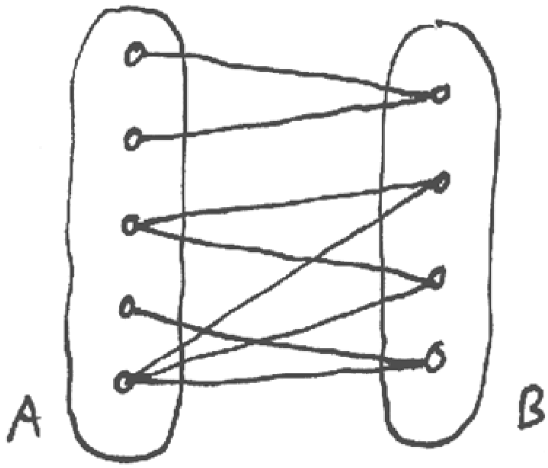
Complete graphs

The *complete graph* K_n is the graph with n vertices where every pair of different vertices is connected by an edge.
(Also called: *Clique*.)

K_1 K_2 K_3 K_4 K_5

Bipartite graphs

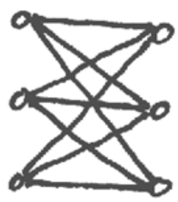
A *bipartite graph* can be split into two disjoint sets of vertices, A and B, such that all edges go from a vertex from A to a vertex from B.



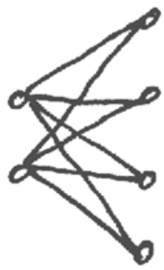
(The edges then form a relation between A and B.)

The complete bipartite graph $K_{m,n}$ is a bipartite graph with $|A| = m$, $|B| = n$, and edges go from every vertex of A to every vertex of B.

$K_{2,3}$



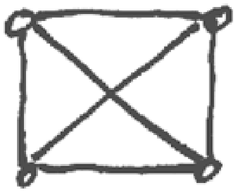
$K_{3,3}$



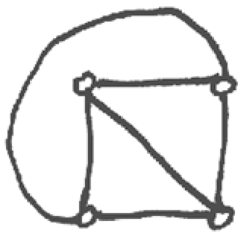
$K_{2,4}$

Planarity

A graph is *planar* if its vertices and edges can be embedded in the plane, with edges as arcs in the plane, such that no two different edges intersect in points different from their start and end vertex.



non-planar embedding

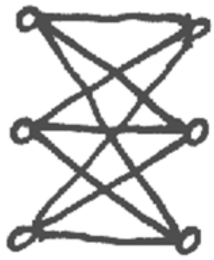


planar embedding of the same graph

Kuratowski's theorem:

A graph is planar if and only if it does not contain any subdivision of K_5 or $K_{3,3}$.

K_5

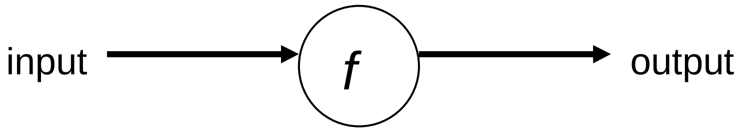


$K_{3,3}$

5. Functions

The *function* is a fundamental notion in mathematics. It is used to describe:

- a dependency between two variables (e.g., between measured sizes of the same objects)
- a transformation of data during some calculation or processing step



- a development of a variable in time or in space (e.g., heigth growth of a plant; magnetic field strength in space...)

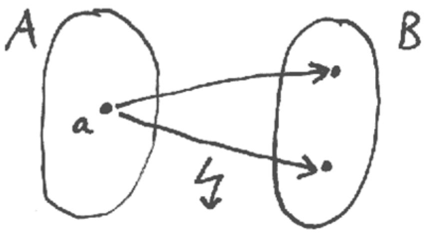
Frequently used synonyms for *function*:
mapping, *transformation*, *operator*

The precise definition of a function identifies it with the relation between "input" (argument(s)) and "output" (value), i.e., a function is defined as a special case of a relation:

A relation R between the sets A (= possible input values) and B (= possible values) is a *function* if for every $a \in A$ there is exactly one $b \in B$ with $a\ R\ b$.
We write then f instead of R and use frequently the notation $f(a) = b$.
Further typical notations:

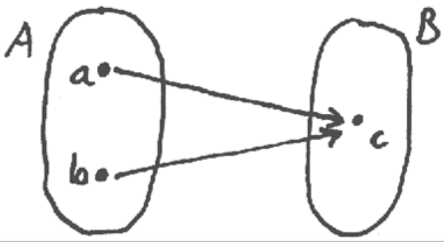
$$f: A \to B, \quad a \mapsto b\ .$$

The following situation is thus excluded for functions, because *a* would have two different "images" in *B*:



f(*a*) must be *unique*.

Allowed is:



f(*a*) = *c*, *f*(b) = *c* .

Written as set: $f = \{(a, c); (b, c)\} \subseteq A \times B$.

We say: "*f* maps *a* to *c*", "*c* is an image of *a* under *f* ".

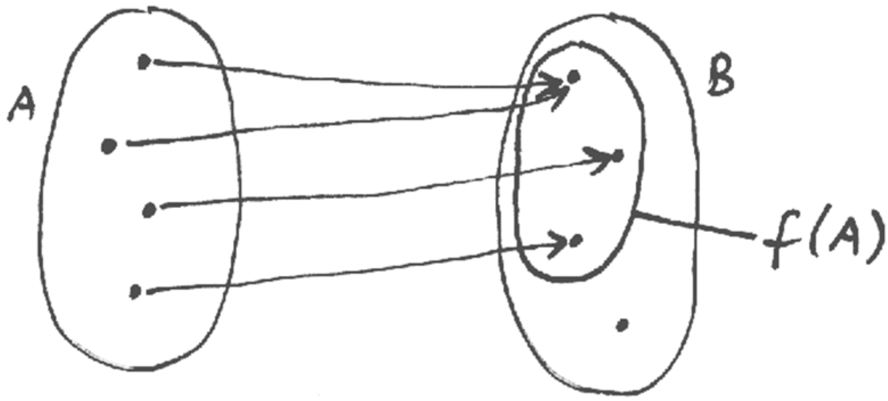
f is the function, *f*(*a*) is a special value.
a is called the *argument* of *f*.

Different notations:

f(*a*) or *fa*	prefix notation
af	postfix notation

Domain and image of a function

$f : A \to B$



A is called the *domain* of *f*
f(*A*) is called the *image* of *A* under *f*,
sometimes also the *range* of *f*.

Multivariate functions

Functions can have several arguments:

$$f: A \times B \to C$$

$$(a, b) \mapsto f(a, b) = c \in C$$

$$a \in A \quad b \in B$$

$$f: A_1 \times \ldots \times A_n \ni (a_1, \ldots, a_n) \mapsto f(a_1, \ldots, a_n) \in C$$

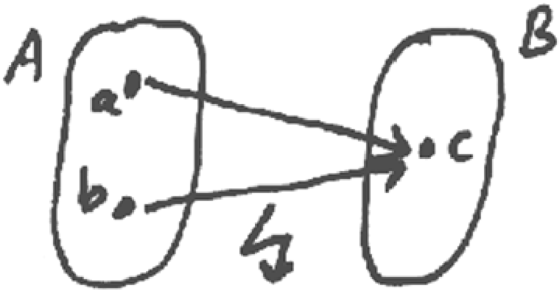
Injective, surjective, bijective functions

Injectivity

A function $f: A \to B$ is called *injective* if

$$\forall a, b \in A: \ a \neq b \ \Rightarrow \ f(a) \neq f(b).$$

That means, two different elements of *A* have always different images. Not allowed is:

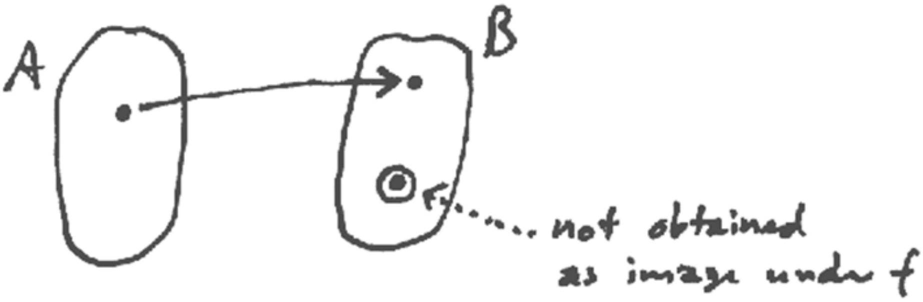


Surjectivity

A function $f: A \to B$ is called *surjective* if

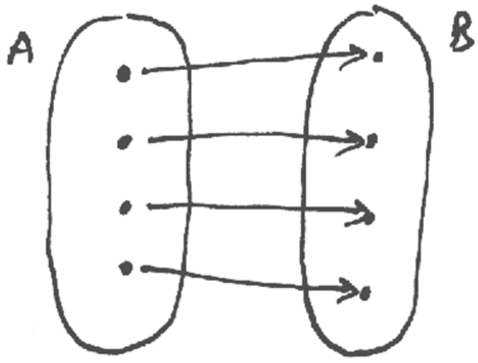
$$\forall b \in B \ \exists a \in A: f(a) = b.$$

All elements of *B* are thus images of elements of *A*. Not allowed is:



Bijectivity

$f: A \to B$ is called *bijective* if it is injective and surjective.



Bijective functions can be *inverted*,
i.e., the converse relation $f^{-1}: B \to A$ is again a function.
That means: $f^{-1}(b)$ is *unique* for every $b \in B$.

An example where this is not the case:

$f(x) = x^2 \quad A = B = \mathrm{IR}$

$f(2) = 4$

$\Rightarrow f^{-1}(4)$ not unique, f^{-1} not a function

$f(-2) = 4$

f is not bijective on IR.

How to obtain the inverse function of a bijective real-valued function (with one argument):

- solve $f(x) = y$ for x, so you obtain $x = f^{-1}(y)$
- switch the names of the variables ($x \leftrightarrow y$).