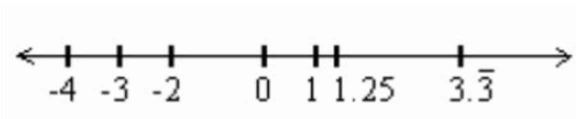
6. Numbers

We do not give an axiomatic definition of the *real numbers* here.
Intuitive meaning: Each point on the (infinite) line of numbers corresponds to a real number, i.e., an element of $\mathbb{R}$.

The line of numbers:



Important subsets of $\mathbb{R}$:

$\mathbb{N}$ the set of all natural numbers (positive integers), does not contain the 0
$\mathbb{N}_0 := \mathbb{N} \cup \{0\}$ the set of all non-negative integers
$\mathbb{Z}$ the set of all integers { ... –2; –1; 0; 1; 2; ... }
$\mathbb{Q}$ the set of all rational numbers (representable as fractions of integers p/q, where $q \neq 0$)

We have:
$\mathbb{N} \subseteq \mathbb{Z} \subseteq \mathbb{Q} \subseteq \mathbb{R}$.

Remark:
Every rational number can be represented as decimal number with its expansion after the decimal dot either coming to an end or becoming periodic.

Examples:

$1/4 = 0.25$

$1/7 = 0.\overline{142857}$ (periodic)

$1/6 = 0.1\overline{6}$ (ultimately periodic)

Example for a transformation in the other direction:

$$0,\overline{62} = 0,62 \cdot (1 + \frac{1}{100} + \frac{1}{10000} + \frac{1}{1000000} + \ldots) = 0,62 \cdot \frac{100}{99} = \frac{62}{99}$$

(note the different notations:
decimal dot in anglosaxon countries, comma in Germany)

Irrational numbers are real numbers that are not rational, i.e., cannot be expressed as a fraction of integers.
Their decimal expansion becomes never periodic.

Examples:

$$\sqrt{2} = 1{,}41421\ 35623\ 73095\ 04880\ 16887\ 24209\ 69807\ 85696\ 71875\ 37694\ \ldots$$

$$\pi = 3{,}14159\ 26535\ 89793\ 23846\ 26433\ 83279\ 50288\ 41971\ 69399\ 37510\ \ldots$$

$$e = 2{,}71828\ 18284\ 59045\ 23536\ 02874\ 71352\ 66249\ 77572\ 47093\ 69995\ \ldots$$

Arithmetic operations on IR:

Addition

Operation symbol: +

$a + b$ exists for every $a, b \in$ IR.
+ can be seen as a function with two arguments:
$a + b$ is in prefix notation $+(a, b)$.

Rules for adding numbers:

$a + b = b + a$ (*commutativity*)

$(a + b) + c = a + (b + c)$ (*associativity*)

$a + 0 = a$ (0 is the *neutral element* of addition)

For every a, there is a number $–a$ such that $a + (–a) = 0$

We have always: $–(–a) = a$.

Subtraction can be derived from addition: $a – b = a + (–b)$.

Multiplication

Operation symbol: $\cdot$ (often omitted!) (sometimes also $*$ instead of $\cdot$).

$a \cdot b$ exists for every $a, b \in \mathbb{R}$.

Rules for multiplication:

$a \cdot b = b \cdot a$

$(a \cdot b) \cdot c = a \cdot (b \cdot c)$

$a \cdot 1 = a$ (1 is the *neutral element* of multiplication)

Rule combining addition and multiplication:

$a \cdot (b + c) = a \cdot b + a \cdot c$ (*distributivity*)

Note: By convention, $\cdot$ binds stronger than +

For every $a \neq 0$, there is a number $1/a$ such that $a \cdot 1/a = 1$.

We have always: $1/(1/a) = a$.

Other notations for $1/a$: $\frac{1}{a}$, $a^{–1}$.

$1/a$ is called the *inverse* of a.
Division can be derived from multiplication:
$a : b = a \cdot 1/b$

Another notation for $a : b$ is $\frac{a}{b}$.
$a : b$ is not defined for $b = 0$.

The power of a number

A power with a positive integer exponent is defined as an iterated multiplication:
Example: $4^3 = 4 \cdot 4 \cdot 4$.
4 is called the basis, 3 the exponent.

By definition, $a^0 = 1$ for all $a \neq 0$.
For $n > 0$, we define as the power with negative exponent $-n$: $a^{-n} = 1/(a^n)$ ($= (a^n)^{-1}$).
Example: $4^{-3} = (4^3)^{-1} = \frac{1}{4^3} = \frac{1}{64}$

The root of a number

For every positive real number a and every positive integer n there exists a positive real number x which fulfills the equation $x^n = a$.
This (unique) x is called the n-th root of a.
Two notations for x:

$$\sqrt[n]{a} = a^{\frac{1}{n}} \quad \forall\, a \in \mathbb{R}$$

For *odd* integers n and negative a we can extend this definition by $a^{1/n} = -(-a)^{1/n}$.

For *even n*, the *n*-th root of a negative number is not defined in IR.

To overcome this restriction, it is possible to extend the set of real numbers IR:

The so-called imaginary unit $i = \sqrt{-1}$ is defined which fulfills

$$i \cdot i = -1.$$

IR is extended to the set C of *complex numbers*.

Each complex number has the form $a + b \cdot i$ with $a, b \in$ IR.

It is possible to calculate with complex numbers in the same way as with real numbers.

Visualization as points in the plane (with real-valued coordinates *a*, *b*).

Back to the real numbers:

The operation "*n*-th root of..." does invert the power operation.

Attention:

We have (by definition) $(\sqrt{x})^2 = x$,

but: $\sqrt{x^2} = |x|$!

Here, $|x|$ denotes the *absolute value* of *x*:

$|x| = x$ if $x \geq 0$ and $|x| = -x$ otherwise.

$|a - b|$: the *distance* between *a* and *b*.

In the context of square roots, the solution formula for quadratic equations ("*pq formula*") is often a useful tool:

For the equation $x^2 + px + q = 0$,
the solutions (if they exist) are:

$$x_{1,2} = \frac{-p}{2} \pm \sqrt{\frac{p^2}{4} - q}$$

Condition for the existence of the solution(s):

$$\frac{p^2}{4} - q \geq 0$$

For control purposes, Vieta's theorem can be useful:
The two solutions fulfill $x_1 + x_2 = -p$
and $x_1 \cdot x_2 = q$.

The power of real numbers with rational exponent:

The power $a^{k/n}$ is defined as

$$a^{\frac{k}{n}} = \sqrt[n]{a^k}$$.

(By using limits of series of rational numbers – for the introduction of limits see later – , the definition of a power can also be extended to irrational exponents.)

Rules for powers:

$a^r \cdot a^s = a^{r+s}$
$a^r : a^s = a^{r-s}$
$(a^r)^s = a^{rs}$
$a^r \cdot b^r = (a \cdot b)^r$

Because the power operation a^n is not commutative, there are two different reverse operations: You can search for a basis or you can search for an exponent. The first case leads to the root, the second case to the *logarithm*.

Definition:
Let a, $b > 0$ be real numbers. The (unique) solution of $b^x = a$ is $x = \log_b a$ (logarithm of a to the base b).

Often the so-called *natural logarithm* is used, which uses the Euler number $e = 2.718281828...$ as its base: $\ln a = \log_e a$.
Other frequent cases: binary logarithm (base 2); decimal logarithm (base 10).

In general, we have: $\log_b a = \ln a / \ln b$.

Rules for logarithms (hold for arbitrary base):

$\log(x \cdot y) = \log x + \log y$
$\log (x / y) = \log x - \log y$
$\log (x^y) = y \cdot \log x$

$$\log\left(\sqrt[n]{x}\right) = \frac{1}{n} \cdot \log x$$

The order relation on $\mathbb{R}$

Every two real numbers a, b can be ordered:

Either $a < b$, or $a = b$, or $a > b$.

$a \leq b$ means $a < b$ or $a = b$.

We have:

$a < b \Rightarrow a + c < b + c$ (analogously for $\leq$),

for $c > 0$: $a < b \Rightarrow a \cdot c < b \cdot c$

<u>but</u> for $c < 0$: $a < b \Rightarrow a \cdot c > b \cdot c$

Bounded intervals

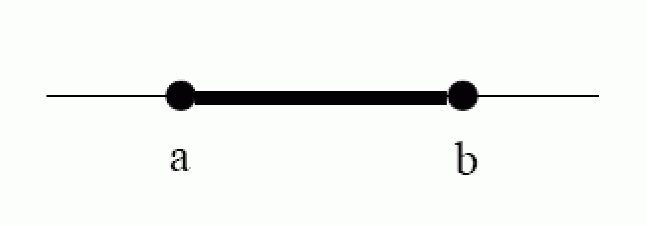
An *open, bounded interval* (a, b) is the set of all real numbers x which are *properly between a* and b, i.e., which fulfill $a < x < b$.

Attention! The same notation as for ordered pairs is used, but the meaning is different.

If $a < b$, (a, b) is an infinite set.

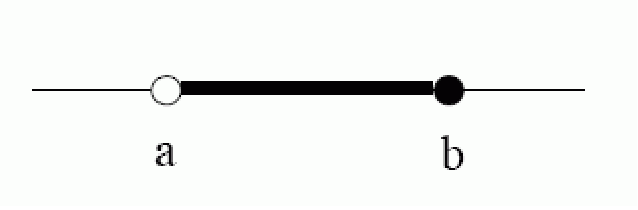
(a, b)

In a *closed interval* $[a, b]$, the end points are included: $[a, b] = \{ x \in \mathbb{R} \mid a \leq x \leq b \}$.



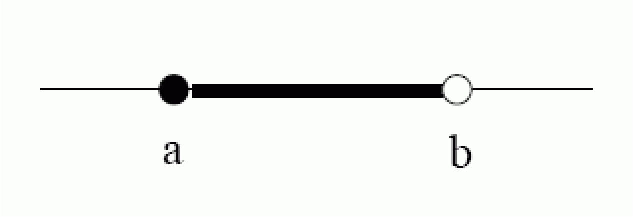
$[a, b]$

An interval closed on the right-hand side:



$(a, b]$

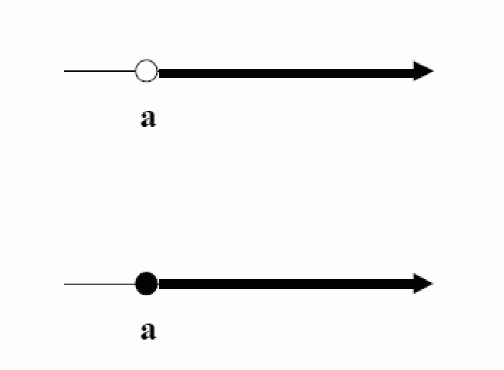
An interval closed on the left-hand side:



$[a, b)$

Unbounded intervals

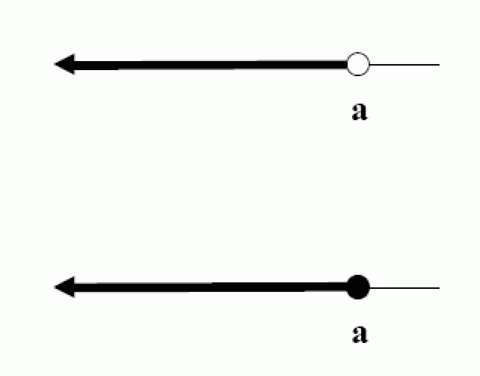
$(a, +\infty) = \{ x \in \mathbb{R} \mid a < x \}$.



$(a, +\infty)$

$[a, +\infty)$

analogously for intervals unbounded to the left:



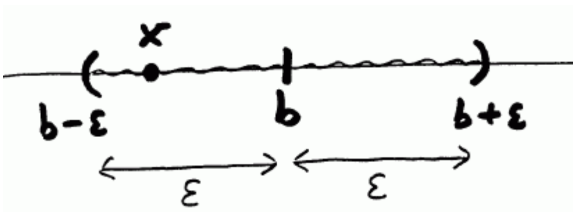
$(-\infty, a)$

$(-\infty, a]$

The *neighbourhood* of a number

Let $\varepsilon > 0$ be a positive real number.
The interval $(b - \varepsilon, b + \varepsilon)$ is called the
ε-neighbourhood of the number b.

We have $(b - \varepsilon, b + \varepsilon) = \{ x \in \mathbb{R} \mid |x - b| < \varepsilon \}$.
That means: The neighbourhood contains all numbers for which the *distance* to b is smaller than the given threshold ε.



Bounds

An *upper bound* of a set M of real numbers is a number r with $r > x$ for all $x \in M$.

Analogously: *lower bound* (exchange > by <).

A set of numbers is called *bounded* if there exists an upper bound and a lower bound for it.

If a set has an upper bound, it has infinitely many upper bounds. We are interested in the smallest one:

The smallest upper bound of a set $M \subseteq \mathbb{R}$ is called the *supremum* of M, denoted sup M.

Analogously:
The largest lower bound of a set $M \subseteq \mathbb{R}$ is called the *infimum* of M, denoted inf M.

Examples:

inf {1; 2; 3; 4} = 1, sup {1; 2; 3; 4} = 4,

$$inf\left\{\frac{1}{n} \mid n \in \mathbb{N}\right\} = 0$$

<u>7. Vectors</u>

We will work with elements from the set

$$\mathbb{R}^n = \mathbb{R} \times \mathbb{R} \times \mathbb{R} \times \ldots \times \mathbb{R}$$

The elements are *n*-tuples of real numbers, we call them *vectors*.

To distinguish vector-valued variables from variables standing for single numbers, often an arrowed letter ($\vec{a}$) or printing in a different font is used.

Two ways to write down a vector:

row vector, e.g., (1; 5; –2)

column vector $\begin{bmatrix} 1 \\ 5 \\ -2 \end{bmatrix}$

To distinguish real numbers from vectors, we call them also *scalars*:

$$\vec{a} = \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{pmatrix} \in \mathbb{R}^n$$ <u>vector</u>

(for n = 2; 3 geometrically:
representation by arrow ; “directed entity“)

$m \in \mathbb{R}$ <u>scalar</u> (“undirected entity“)

$\begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} \cdots$ a_1, a_2, ... are called *components* of the vector

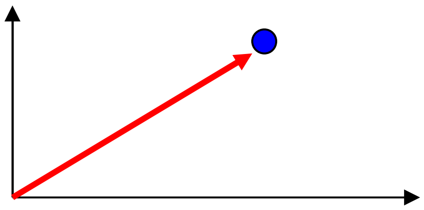
(also: *coordinates*)

special cases:

$\mathbb{R}^1 = \mathbb{R}$

$\mathbb{R}^2 = \mathbb{R} \times \mathbb{R}$, can be represented as a plane:

each vector $\begin{bmatrix} x \\ y \end{bmatrix} \in \mathbb{R}^2$ corresponds to a point in the plane. Often a vector is represented as an arrow pointing from the origin to this point.

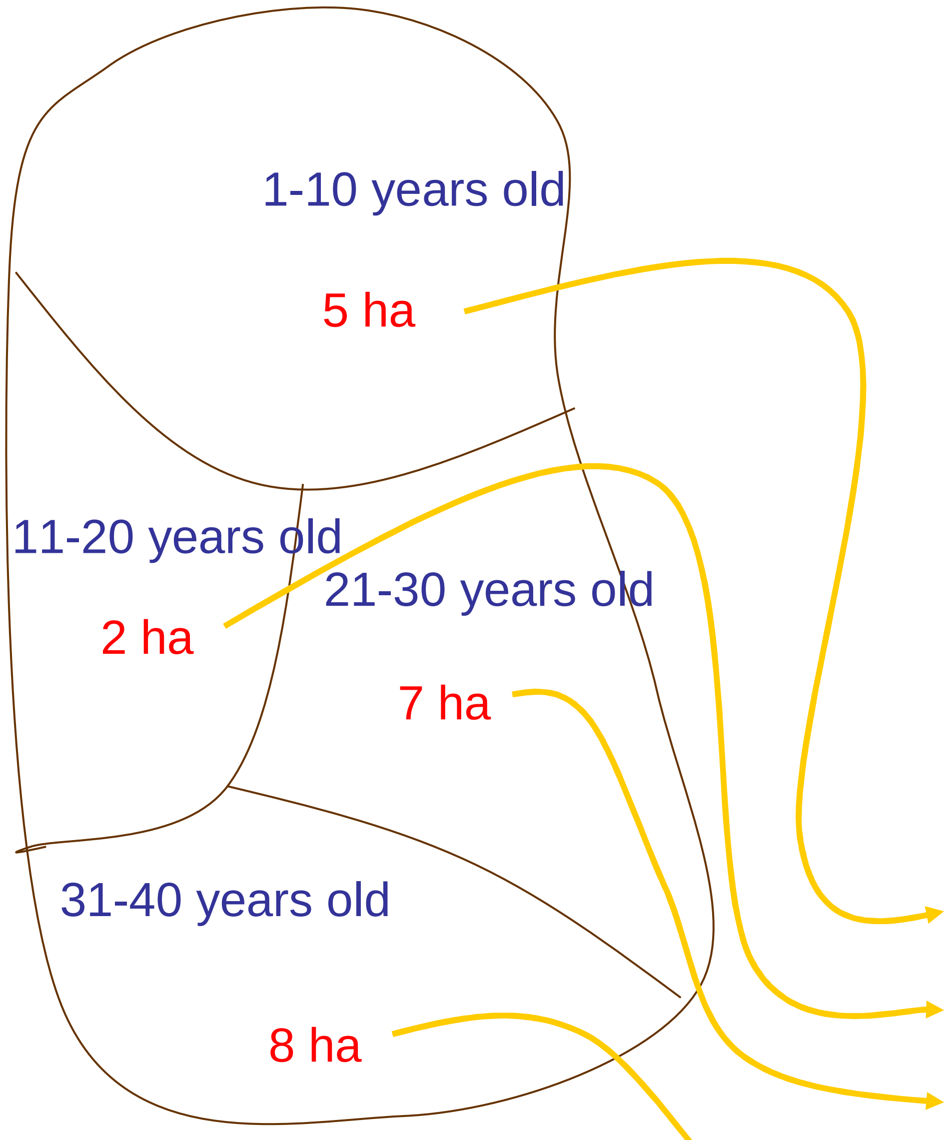


$\mathbb{R}^3 = \mathbb{R} \times \mathbb{R} \times \mathbb{R}$ 3-dimensional space.

$\mathbb{R}^n$ is called an *n-dimensional vector space*.

Example of a vector in a higher-dimensional vector space $\mathbb{R}^n$ ($n > 3$) :

The age-class vector of a population (e.g., of a forest stand)



$$\begin{pmatrix} 5 \\ 2 \\ 7 \\ 8 \end{pmatrix}$$

Equality of vectors:

Two vectors are equal iff all their corresponding components are equal.

$$\begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{pmatrix} \Leftrightarrow \quad a_1 = b_1 \quad \wedge \quad a_2 = b_2 \quad \wedge \cdots \quad \cdots \wedge a_n = b_n$$

Addition of vectors:

Definition of the sum of two vectors in $\mathbb{R}^n$

$$\begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{pmatrix} + \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{pmatrix} = \begin{pmatrix} a_1 + b_1 \\ a_2 + b_2 \\ \vdots \\ a_n + b_n \end{pmatrix}$$

Properties of the addition of vectors:

$\vec{a} + \vec{b} = \vec{b} + \vec{a}$ commutativity

$(\vec{a} + \vec{b}) + \vec{c} = \vec{a} + (\vec{b} + \vec{c})$ associativity

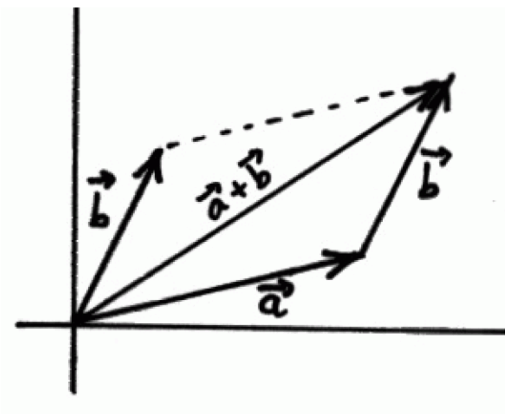
$\vec{a} + \vec{0} = \vec{a},$ neutral element $\vec{0}$

where $\vec{0}$ is the zero vector:

$$\vec{0} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \in \mathrm{IR}^n$$

Geometrical interpretation of vector addition:

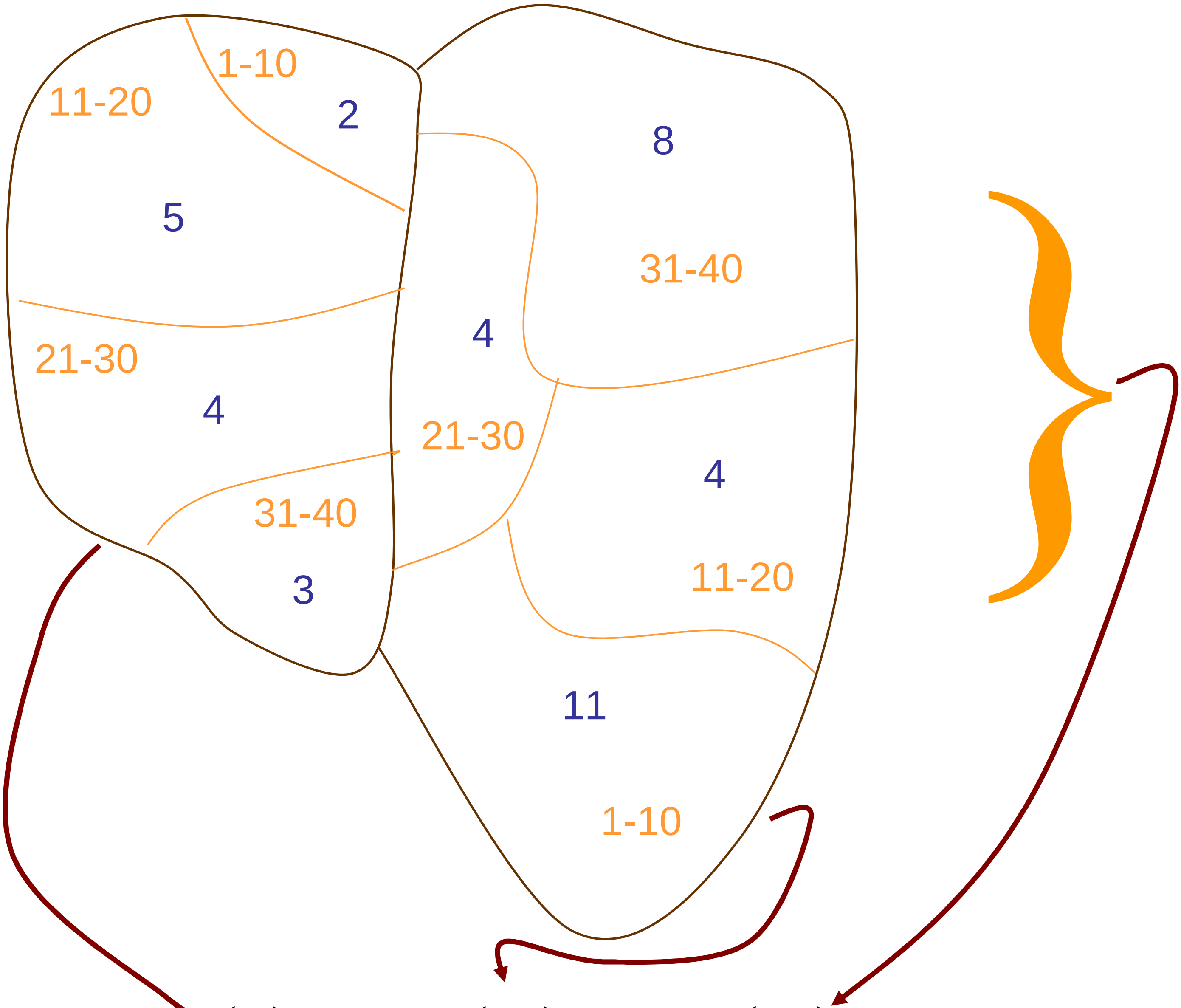
The arrows of both vectors are placed one after the other, and the origin is connected with the new end point.



(in physics: "parallelogram of forces")

The sum in the case of age-class vectors:

aggregation of two forest stands into one.



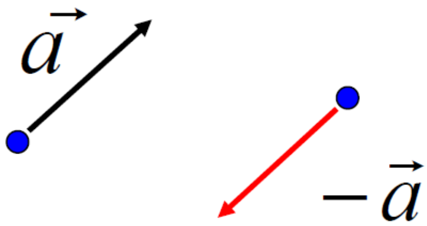
$$\begin{pmatrix} 2 \\ 5 \\ 4 \\ 3 \end{pmatrix} + \begin{pmatrix} 11 \\ 4 \\ 4 \\ 8 \end{pmatrix} = \begin{pmatrix} 13 \\ 9 \\ 8 \\ 11 \end{pmatrix}$$

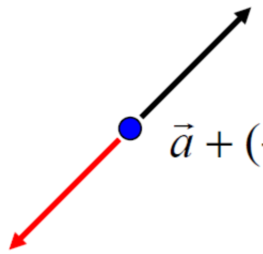
age-class structure of the total area

For all vectors $\vec{a}$ from IR^n, there exists exactly one vector – $\vec{a}$ which fulfills $\vec{a}+(-\vec{a})=\vec{0}$.

↑ inverse (negative) element

$$-\vec{a} \ = \ ?$$





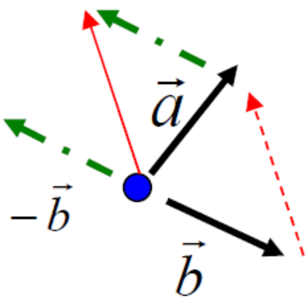
$$\vec{a}+(-\vec{a})=\vec{0}$$

$$=\begin{pmatrix}0\\0\\0\end{pmatrix}$$

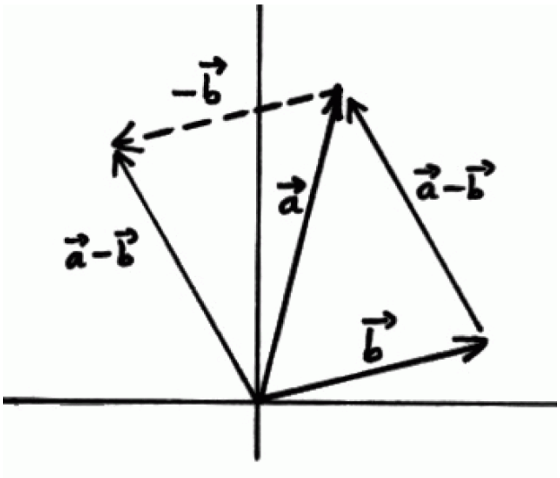
Difference of vectors:

$$\vec{a}-\vec{b}$$

$$=\vec{a}+(-\vec{b})$$ (as in the case of real numbers)



Geometrical interpretation of the difference of vectors:



$$\vec{a}-\vec{b}=\vec{a}+(-\vec{b})$$

$\uparrow$

inversion of the direction

we get thus the "connecting vector" of the endpoints of both vectors.

Multiplication of a vector with a scalar
(≠ „inner product“, ≠ „vector product“ !)

$m \in \mathbb{R}, \quad \vec{a} \in \mathbb{R}^n$

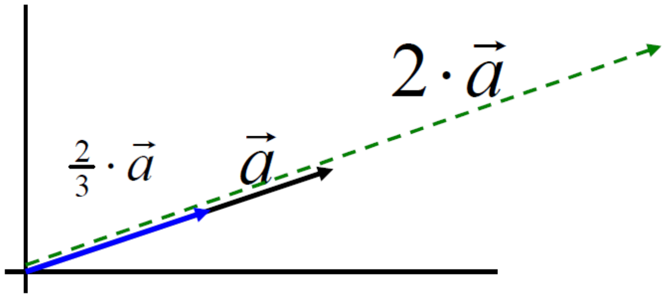
$$m \cdot \vec{a} = m \cdot \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{pmatrix} := \begin{pmatrix} m \cdot a_1 \\ m \cdot a_2 \\ \vdots \\ m \cdot a_n \end{pmatrix} \in \mathbb{R}^n$$

Example:

$$\frac{2}{3} \cdot \begin{pmatrix} 9 \\ -5 \\ 3 \end{pmatrix} = \begin{pmatrix} \frac{2}{3} \cdot 9 \\ \frac{2}{3} \cdot (-5) \\ \frac{2}{3} \cdot 3 \end{pmatrix} = \begin{pmatrix} 6 \\ -\frac{10}{3} \\ 2 \end{pmatrix}$$

geometrical meaning:

expansion, resp. compression of $\vec{a}$ by the factor *m*



The direction is inverted if the factor *m* is < 0.

We have the following rules:

$$1 \cdot \vec{a} = \vec{a}$$

$$0 \cdot \vec{a} = \vec{0}$$

$$(-1) \cdot \vec{a} = -\vec{a}$$

$$m \cdot \vec{0} = \vec{0}$$

$$m \cdot \vec{a} = \vec{0} \quad \Rightarrow \quad m = 0 \vee \vec{a} = \vec{0}$$

$$\left.\begin{array}{l} m \cdot (\vec{a} + \vec{b}) = m \cdot \vec{a} + m \cdot \vec{b} \\ (k+m) \cdot \vec{a} = k \cdot \vec{a} + m \cdot \vec{a} \end{array}\right\} \text{distributive laws}$$

In the following, terms of the form

$$m_1 \cdot \vec{a}_1 + m_2 \cdot \vec{a}_2 + \ldots + m_k \cdot \vec{a}_k$$

$$(= \sum_{i=1}^{k} m_i \cdot \vec{a}_i), \quad m_i \in \mathbb{R}, \quad \vec{a}_i \in \mathbb{R}^n$$

are important. We speak of a linear combination of the vectors $\vec{a}_i, \ldots, \vec{a}_k$; the m_i are called coefficients.

Example (in 3-dimensional space):

$$\vec{a}_1 = (1, -1, 0) \quad , \quad \vec{a}_2 = (2, 1, 1) \quad , \quad \vec{a}_3 = (-2, 0, 0) \quad ,$$

$$\vec{a}_4 = (0, -2, 2)$$

(here written as row vectors for convenience).

The vector

$$\vec{b} = 3\,\vec{a}_1 - 2\,\vec{a}_2 + 0\,\vec{a}_3 + 3\,\vec{a}_4$$

is a linear combination of these four vectors.
In column-vector notation, we calculate:

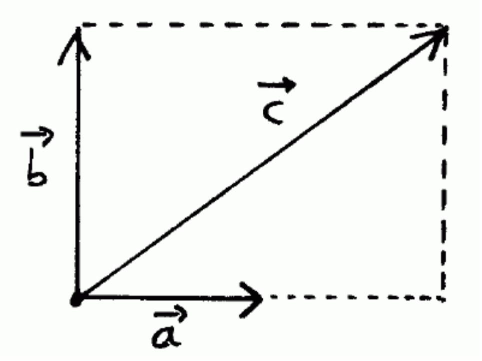
$$\vec{b} \;=\; 3\cdot\begin{bmatrix}1\\-1\\0\end{bmatrix} - 2\cdot\begin{bmatrix}2\\1\\1\end{bmatrix} + 0\cdot\begin{bmatrix}-2\\0\\0\end{bmatrix} + 3\cdot\begin{bmatrix}0\\-2\\2\end{bmatrix} \;=\; \begin{bmatrix}-1\\-11\\4\end{bmatrix}$$

The trivial linear combination

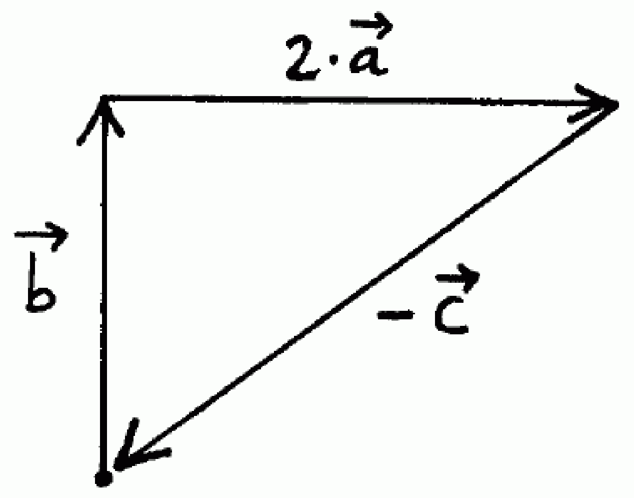
A linear combination is called *trivial* if all coefficients m_1, ..., m_k are 0.
It is called nontrivial if at least one coefficient is *not* 0.

A trivial linear combination has the zero vector as its result.
Can the zero vector also be the result of a *nontrivial* linear combination?

An example: 3 vectors in a plane



We can indeed construct a "cycle" of multiples of these vectors which gives as its sum the zero vector:



$$1\cdot\vec{b}+2\cdot\vec{a}+(-1)\cdot\vec{c}=\vec{0}$$

This is a *nontrivial* linear combination giving the zero vector!
$0\cdot\vec{b}+0\cdot\vec{a}+0\cdot\vec{c}=\vec{0}$ would be trivial.

We say: $\vec{a},\vec{b},\vec{c}$ are *linearly dependent*.

Definition:
Linear dependence / independence of vectors

Given are $k \in \mathbb{N}$ and the vectors
$\vec{a}_1, \vec{a}_2, \ldots, \vec{a}_k \in \mathbb{R}^n$.
These vectors are called *linearly dependent*, if there exist real numbers $m_1, \ldots, m_k$, which are *not all equal to zero*, such that

$$\sum_{i=1}^{k} m_i \vec{a}_i = \vec{0} \ .$$

If the latter equation holds only if all coefficients are 0, then the vectors are called *linearly independent*.

One can prove: Several vectors are linearly dependent if and only if *one of them can be represented as a linear combination of the others*.

Special cases:

$\mathbb{R}^1$: only sets with one element, { *a* }, with $a \neq 0$ are linearly independent.

$\mathbb{R}^2$: $\{\vec{a}_1, \vec{a}_2\}$ is linearly dependent $\Leftrightarrow$ both vectors are on a line through the origin.

$\mathbb{R}^3$: $\{\vec{a}_1, \vec{a}_2, \vec{a}_3\}$ is linearly dependent $\Leftrightarrow$ all three vectors are in a plane going through the origin of the coordinate system.

How to test a set of vectors for linear dependence

Example: Given are the three vectors (1; 2; 3), (0; –1; 0) and (–1; 2; –2). Are they linearly dependent?

Approach: We have to assume $\sum_{i=1}^{3} m_i \vec{a}_i = \vec{0}$.

Written with column vectors, this means:

$$\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} = m_1 \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} + m_2 \begin{bmatrix} 0 \\ -1 \\ 0 \end{bmatrix} + m_3 \begin{bmatrix} -1 \\ 2 \\ -2 \end{bmatrix}$$

For each component, we obtain an equation, giving together the following system of 3 linear equations:

$$\begin{aligned}
0 &= m_1 \qquad\qquad\qquad - m_3 \quad \Rightarrow \quad m_3 = m_1 \\
0 &= 2m_1 \quad - m_2 \quad + 2m_3 \\
0 &= 3m_1 \qquad\qquad\qquad - 2m_3 \quad \Rightarrow \quad 2m_3 = 3m_1
\end{aligned}$$

We can solve this step by step for the unknowns m_i. In this case, we obtain quickly $m_1 = m_2 = m_3 = 0$. So the system can only be fulfilled if all coefficients are zero, and the 3 vectors have been proven als *linearly independent*.

Examples for training:

Linearly dependent or independent? Decide yourself!

(a) $\left\{ \begin{bmatrix} 3 \\ 3 \end{bmatrix}, \begin{bmatrix} -2 \\ -2 \end{bmatrix} \right\}$

(b) $\left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\}$

(c) $\left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right\}$

(d) $\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 5 \\ 4 \\ 0 \end{bmatrix} \right\}$

(e) $\left\{ \begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 3 \\ 3 \end{bmatrix} \right\}$

Rank of a set of vectors

The number of elements of the *maximal* linearly independent subset of a given set of vectors is called the *rank* of the set of vectors.

The *basis* of a vector space

$\mathbb{R}^n$ has infinitely many elements.
Is there a finite subset $\{\vec{a}_1, \ldots, \vec{a}_k\}$,
such that all vectors from $\mathbb{R}^n$
can be represented uniquely as a linear combination of the $\vec{a}_i$?

YES!
Such a set of vectors is called a *basis* of $\mathbb{R}^n$.

Most simple example of a basis:

$$\vec{e}_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \quad \vec{e}_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \quad \ldots, \vec{e}_n = \begin{pmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 1 \end{pmatrix},$$

the *standard basis* of $\mathbb{R}^n$.

There are infinitely many bases, which have, however, all the same number of elements (namely, *n*). This number is called the *dimension* of the vector space.

Example:

$$\left\{ \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 2 \end{pmatrix} \right\}$$ has rank 2

lin. indep. ***lin. dependent***

If we remove $\begin{pmatrix}0\\0\\2\end{pmatrix}$, we obtain a linearly independent vector system:

$\left\{\begin{pmatrix}1\\1\\0\end{pmatrix},\begin{pmatrix}0\\0\\1\end{pmatrix}\right\}$, rank 2.

If we add now, e.g., $\begin{pmatrix}1\\0\\0\end{pmatrix}$, we obtain a basis of IR^3, i.e., a maximal linearly independent subset:

$$\left\{\begin{pmatrix}1\\1\\0\end{pmatrix},\begin{pmatrix}0\\0\\1\end{pmatrix},\begin{pmatrix}1\\0\\0\end{pmatrix}\right\}$$, rank 3.

lin. independent

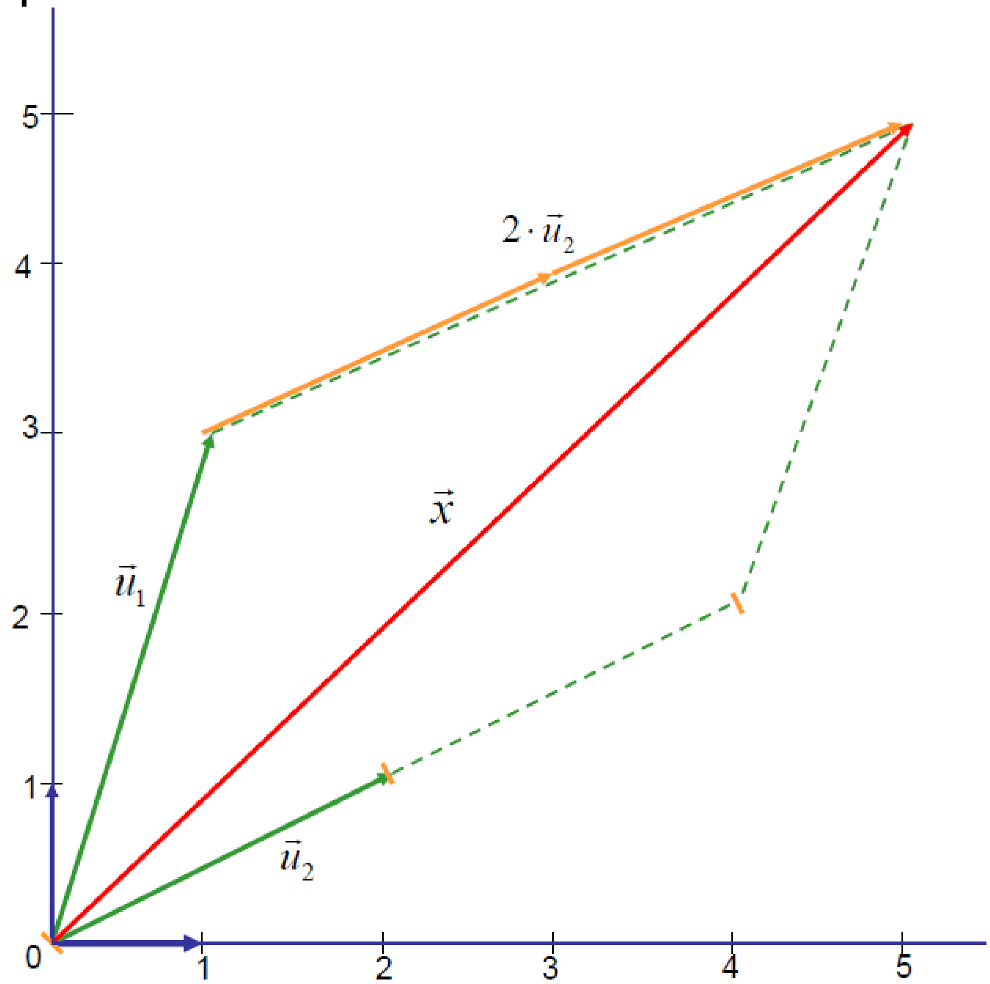
3 is the dimension of IR^3.
If we add an arbitrary further element, e.g., $\begin{pmatrix}0\\1\\0\end{pmatrix}$, the set becomes linearly dependent:

$$1\cdot\begin{pmatrix}1\\1\\0\end{pmatrix}+0\cdot\begin{pmatrix}0\\0\\1\end{pmatrix}+(-1)\cdot\begin{pmatrix}1\\0\\0\end{pmatrix}+(-1)\cdot\begin{pmatrix}0\\1\\0\end{pmatrix}=\vec{0}\ .$$

The *coordinates* of a vector with respect to a given basis

When an arbitrary basis is given, every vector can be expressed uniquely as a linear combination of the elements of this basis (i.e., the coefficients are uniquely determined).

Example:



$$\vec{u}_1 = \begin{pmatrix}1\\3\end{pmatrix},\quad \vec{u}_2 = \begin{pmatrix}2\\1\end{pmatrix},\quad \{\vec{u}_1,\vec{u}_2\}\ \textit{basis of}\ \mathbb{R}^2$$

$$\vec{x} = \begin{pmatrix}5\\5\end{pmatrix} = 1\cdot\vec{u}_1 + 2\cdot\vec{u}_2$$

(1; 2) are the coordinates of $\vec{x}$ w.r.t. $\{\vec{u}_1,\vec{u}_2\}$.

In the special case of the standard basis, we have always:

$$a_1 \cdot \vec{e}_1 + a_2 \cdot \vec{e}_2 + \ldots + a_n \cdot \vec{e}_n$$

$$= a_1 \cdot \begin{pmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix} + a_2 \cdot \begin{pmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} + \ldots + a_n \cdot \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix}$$

$$= \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{pmatrix}$$

The <u>components</u> $a_1, \ldots, a_n$ of a vector $\vec{a} \in \mathrm{IR}^n$ are exactly the <u>coordinates</u> of $\vec{a}$ with respect to the standard basis.