Inner product and cross product

(a) The *inner product* of vectors and the *norm* of a vector

The inner product of two vectors

↓

a product of vectors which gives as result a scalar!

Let there be given: $\vec{x} = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}, \vec{y} = \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{pmatrix} \in \mathrm{IR}^n.$

We define:

$$\vec{x} \cdot \vec{y} \quad := \quad x_1 \cdot y_1 + x_2 \cdot y_2 + \ldots + x_n \cdot y_n$$

$$= \sum_{i=1}^{n} x_i \cdot y_i \in \mathrm{IR}$$

„inner product of $\vec{x}$ and $\vec{y}$“

$\vec{x} \cdot \vec{y}$ is not a vector, thus, e.g., $(\vec{a} \cdot \vec{b}) + \vec{c}$ is senseless.

Example:

$$\begin{pmatrix} 2 \\ 1 \\ 5 \end{pmatrix} \bullet \begin{pmatrix} -1 \\ 3 \\ 8 \end{pmatrix} = 2 \cdot (-1) + 1 \cdot 3 + 5 \cdot 8$$

$$= -2 + 3 + 40$$

$$= 41$$

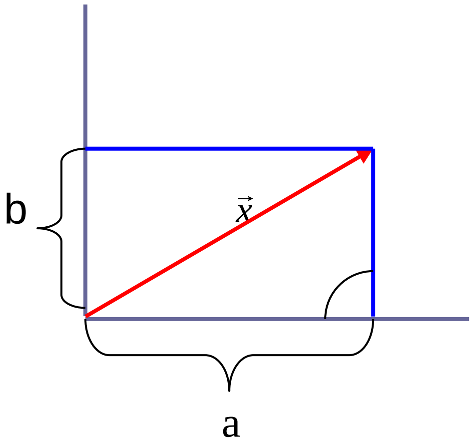
Significance:
The inner product enables propositions about lengths and angles of vectors.

The (Euclidean) *norm* of $\vec{x} \in$ IR^2 is defined as

$$\left\| \begin{pmatrix} a \\ b \end{pmatrix} \right\| = \sqrt{a^2+b^2}$$

= length of $\vec{x}$ according to Pythagoras.

analogously in IR^3.



geometrical interpretation is thus:
norm = length of the vector (arrow).

The vector $\frac{\vec{x}}{\|\vec{x}\|}$ $(i.e. \frac{1}{\|\vec{x}\|} \cdot \vec{x})$ has length 1.
It is called normed.

General definition of the norm (or length) of a vector:

$$\|\vec{x}\| := \sqrt{\vec{x} \cdot \vec{x}} = \sqrt{\sum_{i=1}^{n} x_i^2}$$

Two vectors $\vec{x}, \vec{y}$ are mutually orthogonal (perpendicular) to each other iff $\vec{x} \cdot \vec{y} = 0$.

Example: $$\underset{\text{in } xy \text{ plane}}{\underset{\uparrow}{\begin{pmatrix} 2 \\ 3 \\ 0 \end{pmatrix}}} \cdot \underset{\text{on } z \text{ axis}}{\underset{\nwarrow}{\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}}} = 2 \cdot 0 + 3 \cdot 0 + 0 \cdot 1 = 0$$

Generally, in $\mathrm{I\!R}^n$ the *angle formula* holds:

$$\sphericalangle(\vec{x}, \vec{y}) = \arccos \frac{\vec{x} \cdot \vec{y}}{\| \vec{x} \| \cdot \| \vec{y} \|}$$

(b) The cross product of vectors in $\mathbb{R}^3$

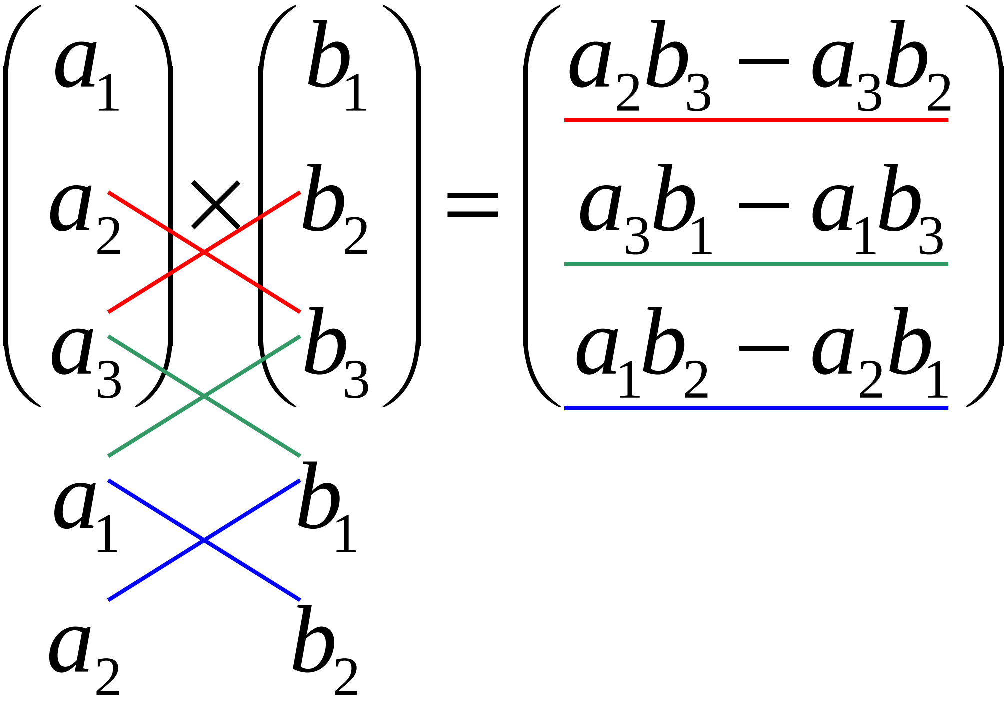
Let there be given two 3-dimensional vectors

$$\vec{a}=\begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix}, \quad \vec{b}=\begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} \in \mathbb{R}^3$$

The *vector product* or *cross product* $\vec{a}\times\vec{b}$ of both vectors is defined as the following new 3-dimensional vector:

$$\vec{a}\times\vec{b}:=\begin{bmatrix} a_2b_3-a_3b_2 \\ a_3b_1-a_1b_3 \\ a_1b_2-a_2b_1 \end{bmatrix} \in \mathbb{R}^3 \quad .$$

Rule for memorizing the components of the cross product:



$$\begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} \times \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix} = \begin{pmatrix} \underline{a_2b_3 - a_3b_2} \\ \underline{a_3b_1 - a_1b_3} \\ \underline{a_1b_2 - a_2b_1} \end{pmatrix}$$

The cross product has the following properties:

$\vec{b}\times\vec{a}=-\vec{a}\times\vec{b}$ (thus, in general, the factors must not be flipped)

$\vec{a}\times\vec{b}=\vec{0} \quad \Leftrightarrow \quad \{\vec{a},\vec{b}\}$ *linearly dependent*

$\vec{a}\times\vec{b}$ stands always *orthogonal* to $\vec{a}$ and $\vec{b}$ (so this is an easy way to find some vector orthogonal to a plane if it is needed)

$\vec{a} \;,\; \vec{b} \;,\; \vec{a}\times\vec{b}$ form in this order a "right-hand system" (orientated like the first three fingers of the right hand)

$\|\vec{a}\times\vec{b}\|=\|\vec{a}\|\cdot\|\vec{b}\|\cdot\sin\sphericalangle(\vec{a},\vec{b})$

= *area of the parallelogram* which is spanned by $\vec{a}$ and $\vec{b}$

Attention:
The cross product does *only* exist in $\mathbb{R}^3$!

8. Linear mappings and matrices

A mapping *f* from IR^n to IR^m is called *linear* if it fulfills the following two properties:

(1) $f(\vec{a}+\vec{b})=f(\vec{a})+f(\vec{b})$ for all $\vec{a},\vec{b}\in\mathrm{IR}^n$

(2) $f(\lambda\vec{a})=\lambda f(\vec{a})$ for all $\lambda\in\mathrm{IR}$ and all $\vec{a}\in\mathrm{IR}^n$

Mappings of this sort appear frequently in the applications. E.g., some important geometrical mappings fall into the class of linear mappings: Rotations around the origin, reflections, projections, scalings, shear mappings...

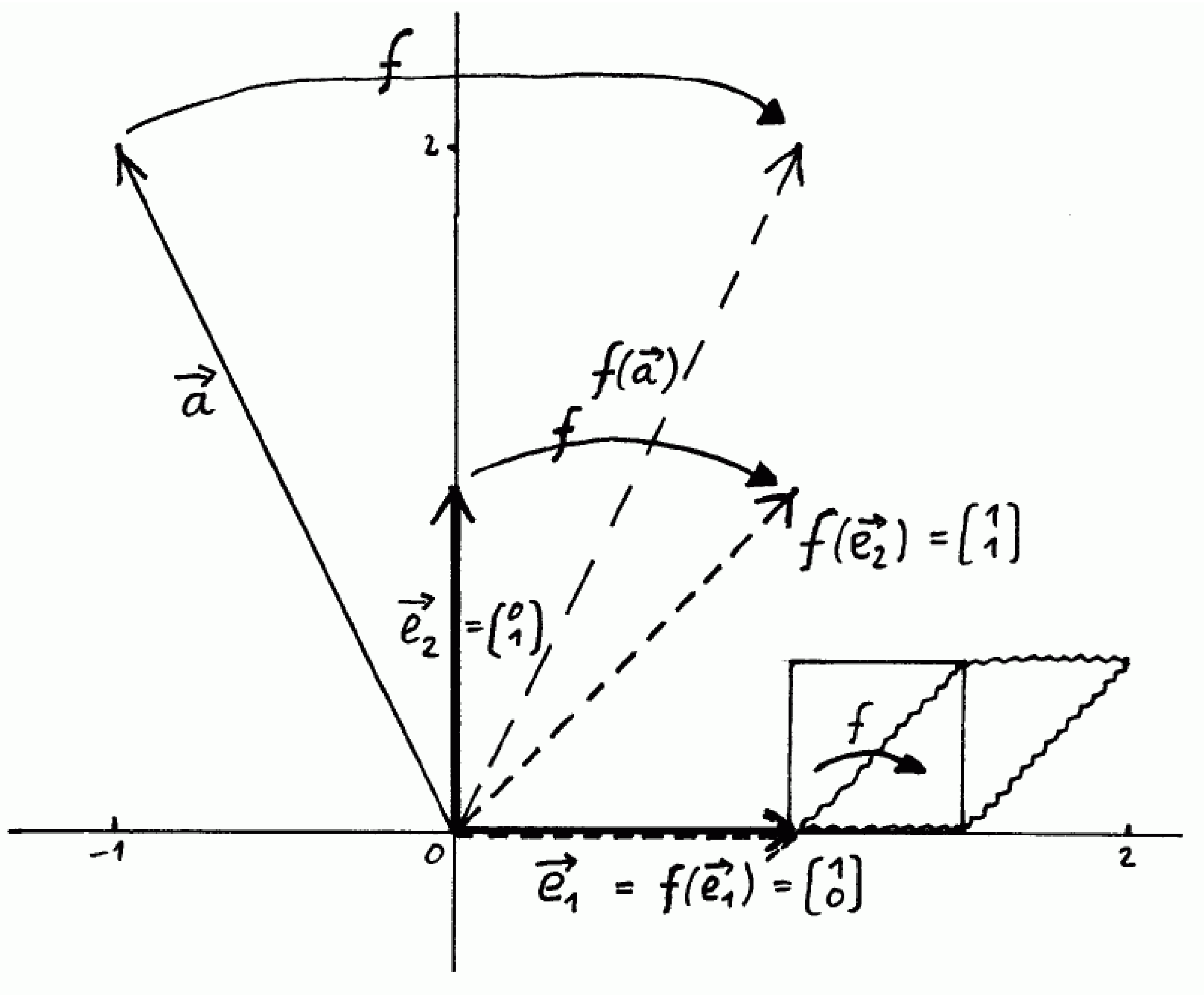
We show at the example of a shear mapping that such a mapping is completely determined (for all input vectors) if its effect on the vectors of the standard basis are known:

Example

Let *f* be the mapping from IR^2 to IR^2 which performs a *shear* along the *x* axis,

i.e., the image of each point under *f* can be found at the same height as the original point, but shifted along the *x* axis by a length which is proportional (in our example: even equal) to the *y* coordinate.

The figure illustrates the effect of *f* at the examples of the standard basis vectors and an arbitrary vector $\vec{a}$:



We have:

$$f : \mathbb{R}^2 \to \mathbb{R}^2$$

$$f\begin{bmatrix}0\\1\end{bmatrix}=\begin{bmatrix}1\\1\end{bmatrix}$$

$$f\begin{bmatrix}0\\2\end{bmatrix}=\begin{bmatrix}2\\2\end{bmatrix}=2\cdot f\begin{bmatrix}0\\1\end{bmatrix}$$

$$f\begin{bmatrix}1\\0\end{bmatrix}=\begin{bmatrix}1\\0\end{bmatrix}$$

f is indeed a linear mapping, that means:

$f(\vec{a}+\vec{b})=f(\vec{a})+f(\vec{b})$ and

$f(c\cdot\vec{a})=c\cdot f(\vec{a})$ are fulfilled.

The general formula for this shear mapping is apparently:

$$f\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x+y \\ y \end{bmatrix}$$

To get knowledge about the image $f\begin{bmatrix} x \\ y \end{bmatrix}$

of an arbitrary vector $\begin{bmatrix} x \\ y \end{bmatrix} \in \mathbb{R}^2$, it is sufficient

to know the *images of the vectors of the standard basis*, i.e., $f\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $f\begin{bmatrix} 0 \\ 1 \end{bmatrix}$:

$$\begin{bmatrix} x \\ y \end{bmatrix} = x\cdot\begin{bmatrix} 1 \\ 0 \end{bmatrix} + y\cdot\begin{bmatrix} 0 \\ 1 \end{bmatrix} = x\cdot\vec{e}_1 + y\cdot\vec{e}_2$$

$$\Rightarrow \quad f\begin{bmatrix} x \\ y \end{bmatrix} = f(x\cdot\vec{e}_1 + y\cdot\vec{e}_2) \overset{\overset{f \text{ is linear}}{\downarrow}}{=} x\cdot f(\vec{e}_1) + y\cdot f(\vec{e}_2)$$

Here: $f\begin{bmatrix} x \\ y \end{bmatrix} = x\cdot\begin{bmatrix} 1 \\ 0 \end{bmatrix} + y\cdot\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} x+y \\ y \end{bmatrix}$,

confirming our formula above.

That means: These images, here $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$,

describe f completely.
They are put together in a *matrix*:

$\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ = matrix of f.

In general:

Matrix of a linear mapping $f: \mathbb{R}^n \rightarrow \mathbb{R}^m$:

$$\begin{array}{cccc}
\left[\begin{matrix}
a_{11} & a_{12} & \cdots & a_{1n} \\
a_{21} & a_{22} & \cdots & a_{2n} \\
\vdots & \vdots & & \vdots \\
a_{m1} & a_{m2} & \cdots & a_{mn}
\end{matrix}\right]
\end{array}$$

$$\begin{array}{cccc}
\uparrow & \uparrow & & \uparrow \\
\text{image of} & \text{image of} & & \text{image of} \\
\begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} & \begin{bmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{bmatrix} & \cdots & \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix}
\end{array}$$

has *m* rows and *n* columns

$\Rightarrow$ "matrix of type (*m*; *n*)"

all entries a_{ij} are real numbers

The matrix describes its associated linear mapping completely.

The result of the application of f to a vector $\vec{x} \in \mathbb{R}^n$ can easily be calculated as the *product* of the *matrix* of f *with the vector* $\vec{x}$.

In our example:

$$f\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 \cdot x + 1 \cdot y \\ 0 \cdot x + 1 \cdot y \end{bmatrix} = \begin{bmatrix} x + y \\ y \end{bmatrix}$$

In the general case:

$$\begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \vdots \\ a_{m1} & \cdots & a_{mn} \end{bmatrix} \cdot \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} a_{11} \cdot x_1 + a_{12} \cdot x_2 + \ldots + a_{1n} \cdot x_n \\ \vdots \\ a_{m1} \cdot x_1 + a_{m2} \cdot x_2 + \ldots + a_{mn} \cdot x_n \end{bmatrix}$$

Example: $\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \cdot \begin{bmatrix} 5 \\ 6 \end{bmatrix} = \begin{bmatrix} 1\cdot 5+2\cdot 6 \\ 3\cdot 5+4\cdot 6 \end{bmatrix} = \begin{bmatrix} 17 \\ 39 \end{bmatrix}$

General definition of a matrix:

A *matrix* of type (m;n), also: $m\times n$ matrix ("m cross n"), is a system of $m \cdot n$ numbers a_{ij}, $i = 1, 2, ..., m$ and $j = 1, ..., n$, ordered in m rows and n columns:

$$\begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix}$$

a_{ij} is called the *element* or *entry* of the i-th row and the j-th column. The $m \cdot n$ numbers a_{ij} are the *components* of the matrix.
A matrix of type (m; n) has m rows and n columns. Each row is an n-dimensional vector (row vector), and each column is an m-dimensional column vector.
The list of elements a_{ii} ($i = 1, 2, ..., r$
with $r = \min(m, n)$) is called the *principal diagonal* of the matrix.

Example:

$$A = \begin{bmatrix} 1 & 4 & -3 & 2 \\ 2 & 3 & 0 & -1 \\ -3 & 4 & 1 & 1 \end{bmatrix}$$

A is of type (3; 4).

A has 3 row vectors:

$\vec{z}_1 = (1, 4, -3, 2)$, $\vec{z}_2 = (2, 3, 0, -1)$, $\vec{z}_3 = (-3, 4, 1, 1)$

and four column vectors:

$$\vec{s}_1 = \begin{bmatrix} 1 \\ 2 \\ -3 \end{bmatrix} , \quad \vec{s}_2 = \begin{bmatrix} 4 \\ 3 \\ 4 \end{bmatrix} , \quad \vec{s}_3 = \begin{bmatrix} -3 \\ 0 \\ 1 \end{bmatrix} , \quad \vec{s}_4 = \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}$$

Its principal diagonal is 1; 3; 1.

Special forms of matrices:

- square matrix:

If $m = n$, i.e., if the matrix *A* has as many rows as it has columns, *A* is called a *square* matrix.

- $m = 1$: A matrix of type (1; *n*) is a row vector.

- $n = 1$: A matrix of type (*m*; 1) is a column vector.

- $m = n = 1$: A matrix of type (1; 1) can be identified with a single real number (i.e., its single entry).

- diagonal matrix:

If *A* is a square matrix and all elements outside the principal diagonal are 0, *A* is called a *diagonal matrix*.

$$\begin{bmatrix} a_{11} & 0 & \cdots & 0 \\ 0 & a_{22} & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & a_{nn} \end{bmatrix}$$

- unit matrix:

The unit matrix *E* is a diagonal matrix where all elements of the principal diagonal are 1.
It plays an important role: Its associated linear mapping is the *identical mapping* $f(\vec{x}) = \vec{x}$.

$$E = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & 1 \end{bmatrix}$$

- zero matrix:

The matrix where *all* entries are 0 is called the zero matrix.

- triangular matrix:

A matrix where all elements below the principal diagonal are 0 is called an *upper triangular matrix*.

Example:

$$A = \begin{bmatrix} 5 & 2 & -1 & 7 \\ 0 & 3 & 1 & 5 \\ 0 & 0 & -1 & 10 \\ 0 & 0 & 0 & 42 \end{bmatrix}$$

Analogous: A matrix where all elements above the principal diagonal are 0 is called a *lower triangular matrix*.

Addition of matrices and *multiplication* of a matrix *with a scalar*:
These operations are defined in the same way as for vectors, i.e., component-wise.

Example:

$$5\cdot\begin{bmatrix}1 & 3\\ 0 & 2\end{bmatrix}+\begin{bmatrix}-1 & 0\\ 7 & 3\end{bmatrix}=\begin{bmatrix}5\cdot 1-1 & 5\cdot 3+0\\ 5\cdot 0+7 & 5\cdot 2+3\end{bmatrix}=\begin{bmatrix}4 & 15\\ 7 & 13\end{bmatrix}$$

Attention: Only matrices of the same type can be added.

Multiplication of a matrix with a column vector: Defined as above, i.e.,

$$\begin{bmatrix}a_{11} & \cdots & a_{1n}\\ \vdots & & \vdots\\ a_{m1} & \cdots & a_{mn}\end{bmatrix}\cdot\begin{bmatrix}x_1\\ \vdots\\ x_n\end{bmatrix}=\begin{bmatrix}a_{11}\cdot x_1+a_{12}\cdot x_2+\ldots+a_{1n}\cdot x_n\\ \vdots\\ a_{m1}\cdot x_1+a_{m2}\cdot x_2+\ldots+a_{mn}\cdot x_n\end{bmatrix}.$$

The result corresponds to the image of the vector under the corresponding linear mapping.

Here, the matrix must have as many columns as the vector has components!

Transposition of a matrix:

Let *A* be a matrix of type (*m*; *n*). The matrix A^{T} of type (*n*; *m*), where its *k*-th row is the *k*-th column of *A* (*k* = 1, ..., *m*), is called the *transposed matrix* of *A*. (Transposition = reflection at the principal diagonal.)

$$\begin{bmatrix}a_{11} & a_{12} & \ldots & a_{1n}\\ a_{21} & a_{22} & \ldots & a_{2n}\\ \vdots & \vdots & & \vdots\\ a_{m1} & a_{m2} & \ldots & a_{mn}\end{bmatrix}^T=\begin{bmatrix}a_{11} & a_{21} & \ldots & a_{m1}\\ a_{12} & a_{22} & \ldots & a_{m2}\\ \vdots & \vdots & & \vdots\\ a_{1n} & a_{2n} & \ldots & a_{mn}\end{bmatrix}$$

Example:

$A = \begin{bmatrix} 1 & 4 \\ -2 & 3 \\ 3 & -1 \end{bmatrix}$ of type (3; 2) $\Rightarrow$

$A^T = \begin{bmatrix} 1 & -2 & 3 \\ 4 & 3 & -1 \end{bmatrix}$ of type (2; 3)

Special case: Transposition of a row vector (type (1; *m*)) gives a column vector (type (*m*; 1)), and vice versa.

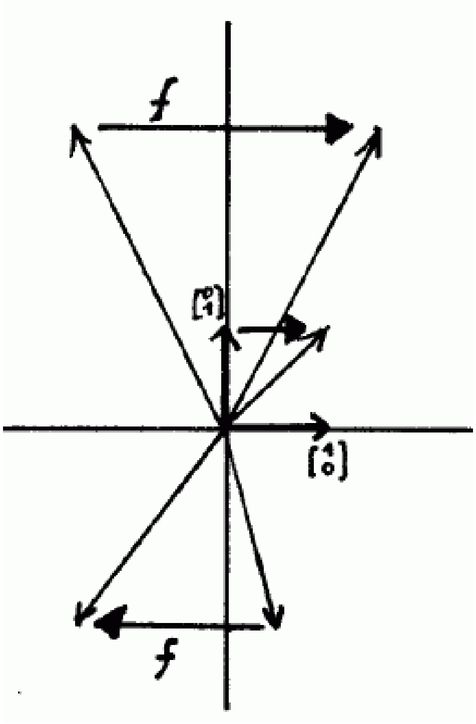
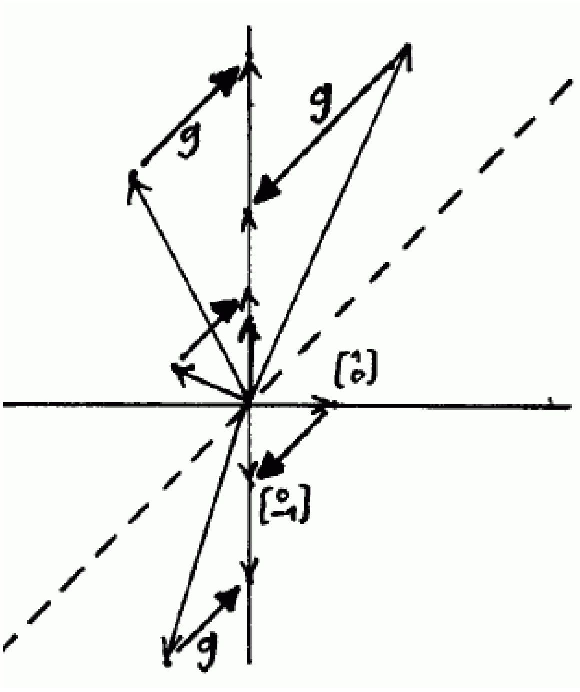
Submatrix:

A *submatrix* of type (*m*–*k*; *n*–*p*) of a matrix *A* of type (*m*; *n*) is obtained by omitting *k* rows and *p* columns from *A*.
The special submatrix derived from *A* by omitting the *i*-th row and the *j*-th column is sometimes denoted A_{ij} .

We now come back to *linear mappings*, which were our entrance point to motivate the introduction of matrices.
Properties of linear mappings are reflected in numerical attributes of their corresponding matrices.

An important example is the so-called *rank* of a linear mapping.

We demonstrate it at two examples:

f : IR^2 $\rightarrow$ IR^2 shear mapping (= example from above)	*g* : IR^2 $\rightarrow$ IR^2 projection along the principal diagonal onto the *y* axis
	
$f\begin{bmatrix}1\\0\end{bmatrix}=\begin{bmatrix}1\\0\end{bmatrix}$ $f\begin{bmatrix}0\\1\end{bmatrix}=\begin{bmatrix}1\\1\end{bmatrix}$	$g\begin{bmatrix}1\\0\end{bmatrix}=\begin{bmatrix}0\\-1\end{bmatrix}$ $g\begin{bmatrix}0\\1\end{bmatrix}=\begin{bmatrix}0\\1\end{bmatrix}$
Matrix of *f* : $\begin{bmatrix}1 & 1\\0 & 1\end{bmatrix}$	Matrix of *g* : $\begin{bmatrix}0 & 0\\-1 & 1\end{bmatrix}$
The images $f\begin{bmatrix}1\\0\end{bmatrix}, f\begin{bmatrix}0\\1\end{bmatrix}$ (i.e., the column vectors of the matrix of *f*) are linearly independent, they span the whole plane IR^2	The images $g\begin{bmatrix}1\\0\end{bmatrix}, g\begin{bmatrix}0\\1\end{bmatrix}$ (i.e., the column vectors of the matrix of *g*) are linearly dependent, they are on the same line through 0 (i.e., on the *y* axis)

⇒ each vector is an image under *f* (*f* is surjective) rank *f* = 2 (= dimension of the plane)	⇒ only the *y* axis is the range of *g* (*g* is not surjective) rank *g* = 1 (= dimension of the line)

Definition:
The <u>rank of a matrix</u> *A* is the maximal number of linearly independent column vectors of *A*.
Notation: *rank* (*A*), r (*A*).

This is consistent with our former definition:
rank (*A*) = rank of the system of column vectors of *A* (as a vector system).

At the same time, it is the dimension of the range of the corresponding linear mapping of *A*.

Theorem:
rank (*A*) is also the maximal number of linearly independent row vectors of *A*.

"column rank = row rank" !

Special cases:
The rank of the zero matrix is 0 (= smallest possible rank of a matrix).
The rank of *E*, the $n \times n$ unit matrix, is *n* (= largest possible rank of an $n \times n$ matrix).

The rank of an $m \times n$ matrix A can be at most the number of rows and at most the number of columns:
$0 \leq rank(A) \leq \min(m, n)$.

For determining the rank of a matrix, it is useful to know that under certain *elementary operations* the *rank* of a matrix *does not change*:

Elementary row operations

(1) Reordering of rows (particularly, switching of two rows)

(2) multiplication of a complete row by a number $c \neq 0$

(3) addition or omission of a row which is a linear combination of other rows

(4) addition of a linear combination of rows to another row.

Analogous for column operations.

Example:

$$A = \begin{bmatrix} 2 & 6 & -4 \\ 3 & 11 & 1 \\ -4 & -14 & 1 \end{bmatrix}$$

By applying elementary row operations, we transform *A* into an upper triangular matrix (parentheses are omitted for convenience):

$$\begin{array}{rrrll}
2 & 6 & -4 & :2 & \\
3 & 11 & 1 & & \\
-4 & -14 & 1 & & \\
\hline
1 & 3 & -2 & \cdot 3 & \cdot 4 \\
3 & 11 & 1 & - & + \\
-4 & -14 & 1 & & \\
\hline
1 & 3 & -2 & & \\
0 & 2 & 7 & + & \\
0 & -2 & -7 & & \\
\hline
1 & 3 & -2 & & \\
0 & 2 & 7 & & \\
0 & 0 & 0 & &
\end{array}$$

The rank of *A* must be the same as the rank of the matrix obtained in the end.

The rank of this triangular matrix can easily seen to be 2 (one zero row; zero rows are always linearly dependent! – The other two rows must be independent because of the first components 1 and 0.)