University of Göttingen
Department Ecoinformatics, Biometrics and Forest Growth
Dr. Irina Kuzyakova, Prof. Dr. Winfried Kurth

Computer Science and Mathematics, summer term 2016

Test exam - Solutions

	Task 1	Task 2	Task 3	Task 4	Task 5	Task 6	Task 7	Task 8	Σ
max.	7	13	5	5	5	5	10	10	60
credits									

Please, read all tasks carefully.
Write down the way how you got your result; credits are given also for the right approach towards a solution.

Time for this exam: 90 min.; no electronic devices are allowed.

- 4 pages in total -

Task 1 (Linear algebra: vectors)

Three vectors in $\mathbb{R}^3$ are given:

$$\vec{a} = \begin{pmatrix} 1 \\ -2 \\ 2 \end{pmatrix},\ \vec{b} = \begin{pmatrix} 6 \\ 4 \\ 1 \end{pmatrix},\ \vec{c} = \begin{pmatrix} -16 \\ 0 \\ -10 \end{pmatrix}.$$

(a) Calculate the vector $2 \cdot \vec{a} + \vec{b}$. *(2 cr.)*

(b) Calculate the inner product $\vec{a} \cdot \vec{b}$. *(1 cr.)*

(c) What is the angle between $\vec{a}$ and $\vec{b}$? *(1 cr.)*

(d) Are the three vectors $\vec{a}$, $\vec{b}$, $\vec{c}$ linearly independent? Prove your answer. *(2 cr.)*

(e) Give a geometrical description of the shape of the set of points in space described by
$\{ \vec{x} \in \mathbb{R}^3 \mid \vec{c} \cdot \vec{x} = 0 \}$
("$\cdot$" denotes the inner product of vectors.) *(1 cr.)*

(a) $2\vec{a} + \vec{b} = \begin{bmatrix} 2\cdot 1 + 6 \\ 2\cdot(-2) + 4 \\ 2\cdot 2 + 1 \end{bmatrix} = \begin{bmatrix} 8 \\ 0 \\ 5 \end{bmatrix}$ (b) $\vec{a}\cdot\vec{b} = 1\cdot 6 + (-2)\cdot 4 + 2\cdot 1 = 0$

(c) $\sphericalangle(\vec{a},\vec{b}) = \arccos \frac{\vec{a}\cdot\vec{b}}{\|\vec{a}\|\cdot\|\vec{b}\|} = \arccos 0 = 90°$ (or $\frac{\pi}{2}$, in radians)

(d) Solution with the determinant:

$$\begin{vmatrix} 1 & 6 & -16 \\ -2 & 4 & 0 \\ 2 & 1 & -10 \end{vmatrix} = 1\cdot 4\cdot(-10) + 0 + (-2)\cdot 1\cdot(-16) - 2\cdot 4\cdot(-16) - 0 - (-10)\cdot(-2)\cdot 6$$

$$= -40 \quad +32 \quad +128 \quad -120$$

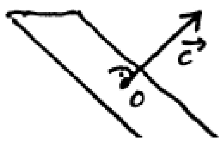
$= 0 \Rightarrow$ the column vectors $\vec{a}, \vec{b}, \vec{c}$ are lin. dependent;

or solve the system $m_1\cdot\vec{a} + m_2\cdot\vec{b} + m_3\cdot\vec{c} = \vec{0}$ and show that it has more than the trivial solution $m_1 = 0, m_2 = 0, m_3 = 0$;

or find such a solution by trial-and-error, e.g.,

$$(+4)\cdot\begin{bmatrix} 1 \\ -2 \\ 2 \end{bmatrix} + (+2)\cdot\begin{bmatrix} 6 \\ 4 \\ 1 \end{bmatrix} + (+1)\cdot\begin{bmatrix} -16 \\ 0 \\ -10 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

(e) $\vec{x}$ fulfills $\vec{c}\cdot\vec{x} = 0$ if and only if $\vec{x}$ is orthogonal to the given vector $\vec{c}$ $\Rightarrow$ the set of points is the plane through O which is orthogonal to $\vec{c}$.



Task 2 (Linear algebra: 2×2 matrices and linear mappings)

The matrix $A = \begin{pmatrix} 0 & 2 \\ 2 & 0 \end{pmatrix}$ is given.

(a) What is the rank of A ? Give a reason for your answer. *(1 cr.)*

(b) Calculate $A \cdot \begin{pmatrix} 1 \\ 2 \end{pmatrix}$ and $A \cdot \begin{pmatrix} 1 \\ -1 \end{pmatrix}$. Draw the vectors $\begin{pmatrix} 1 \\ 2 \end{pmatrix}$, $\begin{pmatrix} 1 \\ -1 \end{pmatrix}$, $A \cdot \begin{pmatrix} 1 \\ 2 \end{pmatrix}$ and $A \cdot \begin{pmatrix} 1 \\ -1 \end{pmatrix}$ in a cartesian coordinate system. *(4 cr.)*

(c) The linear mapping associated with A is $f: \ \vec{x} \mapsto A \cdot \vec{x}$.
Describe (in words) how an arbitrary vector $\vec{x}$ is transformed geometrically by f. *(1 cr.)*

(d) Calculate the matrix A^2. *(1 cr.)*

(e) Calculate $\det A$. *(1 cr.)*

(f) Determine the matrix A^{-1} (if it exists). *(2 cr.)*

(g) Determine the eigenvalues of A. *(3 cr.)*

(a) $\text{rank}(A) = 2$, because $\begin{bmatrix} 0 \\ 2 \end{bmatrix}$ and $\begin{bmatrix} 2 \\ 0 \end{bmatrix}$ are linearly independent; or: because $\begin{vmatrix} 0 & 2 \\ 2 & 0 \end{vmatrix} = -4 \neq 0$.

(b) $\begin{bmatrix} 0 & 2 \\ 2 & 0 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 4 \\ 2 \end{bmatrix}$, $\begin{bmatrix} 0 & 2 \\ 2 & 0 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} -2 \\ 2 \end{bmatrix}$. (See extra figure.)

(c) f mirrors every vector at the principal diagonal and then doubles its length.

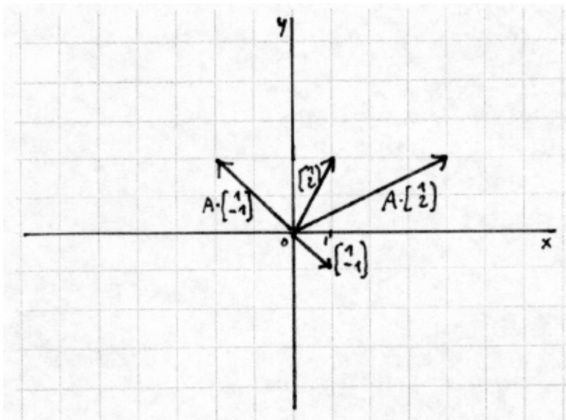
(d) $A^2 = \begin{bmatrix} 0 & 2 \\ 2 & 0 \end{bmatrix} \cdot \begin{bmatrix} 0 & 2 \\ 2 & 0 \end{bmatrix} = \begin{bmatrix} 4 & 0 \\ 0 & 4 \end{bmatrix}$

(e) $\det A = \begin{vmatrix} 0 & 2 \\ 2 & 0 \end{vmatrix} = 0 - 4 = -4$

(f) $A^{-1} = \begin{bmatrix} 0 & \frac{1}{2} \\ \frac{1}{2} & 0 \end{bmatrix}$

(g) $\det(A - \lambda E) = \begin{vmatrix} -\lambda & 2 \\ 2 & -\lambda \end{vmatrix} = \lambda^2 - 4 = (\lambda - 2) \cdot (\lambda + 2) \overset{!}{=} 0 \Leftrightarrow \lambda_1 = -2, \ \lambda_2 = +2$

task 2 (b), graphical part:



Task 3 (Linear algebra: larger matrices and linear systems)

The matrix $A=\begin{pmatrix}2&1&3\\1&1&2\\4&2&6\end{pmatrix}$ is given.

(a) Calculate the product $A\cdot\begin{pmatrix}2&0\\0&1\\-1&0\end{pmatrix}$ *(2 cr.)*

(b) $A\vec{x}=\vec{b}$ with $\vec{b}=\begin{pmatrix}1\\0\\2\end{pmatrix}$ is a system of $m=3$ linear equations for $n=3$ unknowns.

How many unknowns can be chosen arbitrarily? (Give a reason for your answer.) *(3 cr.)*

(a)

$$A\cdot\begin{bmatrix}2&0\\0&1\\-1&0\end{bmatrix}=\begin{bmatrix}2&1&3\\1&1&2\\4&2&6\end{bmatrix}\cdot\begin{bmatrix}2&0\\0&1\\-1&0\end{bmatrix}=\begin{bmatrix}1&1\\0&1\\2&2\end{bmatrix}$$

(b)

$$\overbrace{\underbrace{\begin{array}{ccc}2&1&3\\1&1&2\\4&2&6\end{array}}_{A}\left|\begin{array}{c}1\\0\\2\end{array}\right.}^{A_{ext}}$$

$$\begin{array}{ccc|c|l}
1&1&2&0&\\
2&1&3&1&-2\\
4&2&6&2&-4\\
\hline
1&1&2&0&\\
0&-1&-1&1&\\
0&-2&-2&2&-2\\
\hline
1&1&2&0&\\
0&-1&-1&1&\\
0&0&0&0&
\end{array}$$

$\Rightarrow$ rank A = rank A_{ext} = $2 < 3$ (= nb. of unknowns)

$\Rightarrow$ the linear system has infinitely many solutions (according to Frobenius' theorem), $n-2=1$ unknown can be chosen arbitrarily.

Task 4 (Computer science: programming) *(5 cr.)*

The following Java method **f** gets an integer array **x** and a single integer **a** as its arguments:

```
public int f(int x[], int a)
    {
    int i = 0;
    boolean b = true;
    while (b && (i < x.length))
       {
       if (x[i] == a)
          b = false;
       else
          i = i+1;
       }
    if (b)
       {
       println("Error!");
       return -1;
       }
    else
       return i;
    }
```

What does it give back as its result?

i goes through the positions of array x, starting with 0.

As soon as the corresponding entry of x (x[i]) is equal to a, b gets the value "false" and the while-loop stops.

i is then returned, i.e., the result is the position of the first entry equal to a in array x.

If no "a" is found in x, b remains true, and the value -1 is returned.

Task 5 (Computer science: representation of numbers)

(a) What is the binary representation of the decimal number 63 ? *(1 cr.)*

(b) What is the decimal representation of the hexadecimal number 2A5 ? *(1 cr.)*

(c) Give the 8-bit two's complement of the decimal number –84. *(2 cr.)*

(d) What is the exact value of the fraction which is represented in the ternary system (= base 3) by 0.2222222... (with infinitely many 2s after the dot) ? *(1 cr.)*

(a) $63_{10} = 111111_2$

(b) $2A5_{16} = 2\cdot 256 + 10\cdot 16 + 5\cdot 1 = 677_{10}$

(c) $84_{10} = 01010100_2$ [8 bits]

$$\begin{array}{rl} \rightarrow & 10101011 \\ + & 1 \\ \hline & 10101100 \end{array} \quad \Rightarrow \quad -84 \mathrel{\hat{=}} 10101100$$

(d) $x = 0.2222\ldots_3$

$3x = 2.2222\ldots_3 = 2 + x \quad \Rightarrow \quad 3x - x = 2$

$\Rightarrow \quad 2x = 2$

$\Rightarrow \quad x = 1$

Task 6 (Computer science: rule-based simulation)

(a) Given is the L-system

L1: $A \rightarrow$ [RU(45) F0] F0 A .

Which string is produced after 3 steps of application of this rule to the start word A ? Draw the graphical structure in the plane which is obtained from this string by turtle interpretation. *(2 cr.)*

(b) Now this is modified to the L-system

L2: $A \rightarrow$ [RU(45) F0 A] F0 A .

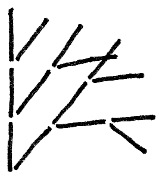
Draw the corresponding graphical structure after 3 steps in this case. *(1 cr.)*

(c) How does the number of single lines (obtained from an "F0" symbol) grow (quantitatively) with the number of steps for L1, how for L2 ? *(2 cr.)*

(a) A → [RU(45) FO] FO A
→ [RU(45) FO] FO [RU(45) FO] FO A
→ [RU(45) FO] FO [RU(45) FO] FO [RU(45) FO] FO A = result after 3 steps

graphically :

(b)



(c) in L1: linearly, in L2: exponentially

(or: in L1: $2n$, in L2: $2^{n+1}-2$, if n is the number of steps.)

Task 7 (Calculus: univariate functions, differentiation) *(10 cr.)*

Given is the following function: $f(x) = \frac{1}{3}x^3 - 4x^2 + 7x - 5$.

(a) Find all x values where the function f has local extrema and classify them as minima or maxima.

(b) Find where the function is increasing / decreasing, and all x values of inflection points.

$$f'(x) = x^2 - 8x + 7$$

$$x^2 - 8x + 7 = 0$$

$$x_{1,2} = \frac{8 \pm \sqrt{64 - 4 \cdot 1 \cdot 7}}{2} = \frac{8 \pm \sqrt{36}}{2}$$

$$f''(x) = 2x - 8$$

$$x_1 = 1; f''(x_1) = 2 \cdot 1 - 8 = -6 < 0 - maximum$$

$$x_2 = 7; f''(x_2) = 2 \cdot 7 - 8 = 6 > 0 - minimum$$

(continued next page)

$$f'(x) = (x-7)(x-1)$$

$$-\infty < x < 1 \ : f'(x) > 0: \ increasing$$

$$1 < x < 7 \ : \ f'(x) < 0: decreasing$$

$$x > 7 : f'(x) > 0 \ increasing$$

$$f''(x) = 0: \quad 2x - 8 = 0 \quad x = 4 \qquad f'''(x) = 2 \neq 0$$

$x = 4$: inflection point

Task 8 (Calculus: integration) *(10 cr.)*

Compute the total area between the function $f(x) = 6x^2 + 6x - 12$, the x axis and the lines $x_1 = 0$ and $x_2 = 2$.

The roots of the function:

$$6x^2 + 6x - 12 = 0$$

$$x_{1,2} = \frac{-6 \pm \sqrt{36 + 4 \cdot 6 \cdot 12}}{12} = \frac{-6 \pm \sqrt{324}}{12}$$

$$x_1 = -2; x_2 = 1$$

$$f(x) = 6x^2 + 6x - 12 = 6(x+2)(x-1)$$

$x < -2$: function positive

$-2 < x < 1$: function negative

$x > 1$: function positive

$$Area = \left| \int_0^1 (6x^2 + 6x - 12)\, dx \right| + \int_1^2 (6x^2 + 6x - 12) dx =$$

$$= |[2x^3 + 3x^2 - 12x]_0^1| + [2x^3 + 3x^2 - 12x]_1^2 =$$

$$= |2 + 3 - 12| + (2 \cdot 2^3 + 3 \cdot 2^2 - 2 \cdot 12) - (2 + 3 - 12) = 7 + 4 - (-7)$$

$$= 18$$