Computer Science and Mathematics: additional exercises

Additional exercises – Solutions

Task 1 (sets)

(a) $B = \{1; 3; 5; 7; 9\}$

(b) $A \cup B = \{1; 2; 3; 5; 7; 9\}$

(c) $A \cap B = \{1; 3\}$

(d) $\mathcal{P}(A) = \{\emptyset; \{1\}; \{2\}; \{3\}; \{1;2\}; \{1;3\}; \{2;3\}; \{1;2;3\}\}$

(e) $A \times C = \{(1;0); (1;1); (2;0); (2;1); (3;0); (3;1)\}$

(f) $C \times C \times C = \{(0;0;0); (0;0;1); (0;1;0); (0;1;1); (1;0;0); (1;0;1); (1;1;0); (1;1;1)\}$

Task 2

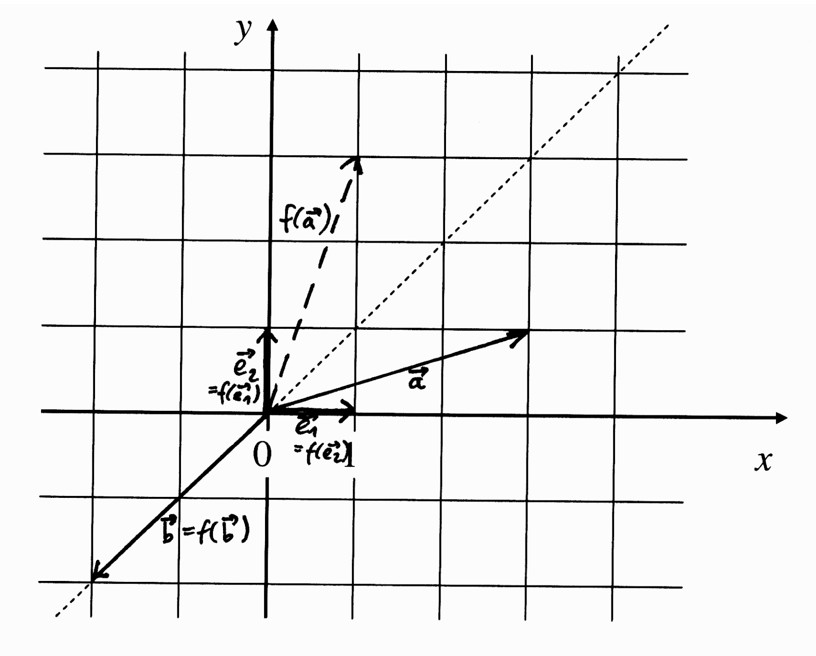
(a) (see extra figure)

$f(\vec{e_1}) = \vec{e_2}$, $f(\vec{e_2}) = \vec{e_1}$, $f(\vec{b}) = \vec{b}$

(b) $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$

(c) $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = E$ (unit matrix)

(d) Applying the mirror transformation twice has the same effect as doing nothing (= applying the unit matrix)



Task 3

(a) $\vec{AB} = B - A = \begin{bmatrix} 3 \\ -2 \\ 0 \end{bmatrix}$, $\vec{BC} = C - B = \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}$, $\vec{AC} = C - A = \begin{bmatrix} 3 \\ -1 \\ -1 \end{bmatrix}$

(b) $\| \vec{AB} \| = \sqrt{3^2 + (-2)^2 + 0^2} = \sqrt{13} \; (\approx 3.606)$

$\| \vec{BC} \| = \sqrt{0^2 + 1^2 + (-1)^2} = \sqrt{2} \; (\approx 1.414)$

$\| \vec{AC} \| = \sqrt{3^2 + (-1)^2 + (-1)^2} = \sqrt{11} \; (\approx 3.317)$

(c) $\cos \sphericalangle(\vec{AB}, \vec{AC}) = \dfrac{\vec{AB} \cdot \vec{AC}}{\|\vec{AB}\| \cdot \|\vec{AC}\|} = \dfrac{3 \cdot 3 + (-2) \cdot (-1) + 0 \cdot (-1)}{\sqrt{13} \cdot \sqrt{11}} = \dfrac{11}{\sqrt{13} \cdot \sqrt{11}} = \dfrac{\sqrt{11}}{\sqrt{13}}$

(≈ 0.920)

(d) area (triangle) $= \frac{1}{2}$ area (parallelogram)

$$= \frac{1}{2} \| \vec{AB} \times \vec{AC} \| = \frac{1}{2} \left\| \begin{bmatrix} 3 \\ -2 \\ 0 \end{bmatrix} \times \begin{bmatrix} 3 \\ -1 \\ -1 \end{bmatrix} \right\|$$

$$= \frac{1}{2} \left\| \begin{bmatrix} 2 \\ 3 \\ 3 \end{bmatrix} \right\| = \frac{1}{2} \sqrt{4+9+9} = \frac{1}{2} \sqrt{22} \; (\approx 2.345)$$

Task 4

(a) $\begin{vmatrix} 1 & 4 & 3 \\ 0 & 2 & 1 \\ 0 & 7 & 4 \end{vmatrix} = 1 \cdot \begin{vmatrix} 2 & 1 \\ 7 & 4 \end{vmatrix} + 0 + 0 = 8 - 7 = 1$

$\begin{vmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 5 \end{vmatrix} = 2 \cdot 3 \cdot 5 = 30$

(b) $\begin{bmatrix} 1 & 4 & 3 \\ 0 & 2 & 1 \\ 0 & 7 & 4 \end{bmatrix} \cdot \begin{bmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 5 \end{bmatrix} = \begin{bmatrix} 2 & 12 & 15 \\ 0 & 6 & 5 \\ 0 & 21 & 20 \end{bmatrix}$

(c)
$$\begin{array}{ccc|ccc|l}
1 & 4 & 3 & 1 & 0 & 0 & \\
0 & 2 & 1 & 0 & 1 & 0 & -2 \quad :2 \\
0 & 7 & 4 & 0 & 0 & 1 & \\
\hline
1 & 0 & 1 & 1 & -2 & 0 & \\
0 & 1 & \frac{1}{2} & 0 & \frac{1}{2} & 0 & -7 \\
0 & 7 & 4 & 0 & 0 & 1 & \\
\hline
1 & 0 & 1 & 1 & -2 & 0 & \\
0 & 1 & \frac{1}{2} & 0 & \frac{1}{2} & 0 & -1 \quad -2 \\
0 & 0 & \frac{1}{2} & 0 & -\frac{7}{2} & 1 & \cdot 2 \\
\hline
1 & 0 & 0 & 1 & 5 & -2 & \\
0 & 1 & 0 & 0 & 4 & -1 & \\
0 & 0 & 1 & 0 & -7 & 2 &
\end{array}$$

$\Rightarrow A^{-1} = \begin{bmatrix} 1 & 5 & -2 \\ 0 & 4 & -1 \\ 0 & -7 & 2 \end{bmatrix}.$

Task 5

Matrix of the system:

$$\underbrace{\underbrace{\left[\begin{array}{cccc} 1 & \frac{1}{2} & 17 & -3 \\ 0 & 2 & 5 & 22 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 2 & 6 \end{array}\right.}_{A}\left.\begin{array}{|c} 0 \\ 5 \\ 4 \\ 8 \end{array}\right]}_{A_{ext}} \xrightarrow{\cdot(-2)} \left[\begin{array}{cccc|c} 1 & \frac{1}{2} & 17 & -3 & 0 \\ 0 & 2 & 5 & 22 & 5 \\ 0 & 0 & 1 & 3 & 4 \\ \hline 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

upper tri-angular

← zero row

$\Downarrow$

$\mathrm{rank}(A) = \mathrm{rank}(A_{ext}) = 3$

$<$ nb. of unknowns $= 4$

$\Downarrow$

the system has infinitely many solutions
(according to Frobenius' theorem)

Task 6

(a) $\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 0 & 1 & 1 \end{bmatrix} \cdot \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 200 \\ 320 \\ 80 \end{bmatrix} \iff \begin{cases} x_1 + x_2 + x_3 = 200 & (1) \\ x_1 + 2x_2 + 3x_3 = 320 & (2) \\ x_2 + x_3 = 80 & (3) \end{cases}$

(b) Eq. (3) $\Rightarrow x_3 = 80 - x_2$

$\hookrightarrow (1) \Rightarrow x_1 + x_2 + (80 - x_2) = 200 \Rightarrow x_1 = 120$

$\hookrightarrow (2) \Rightarrow 120 + 2x_2 + 3(80 - x_2) = 320 \Rightarrow -x_2 = 320 - 120 - 240 = -40$

$\Rightarrow x_2 = 40$

$\Rightarrow x_3 = 80 - 40 = 40$

(c) The system has a single solution $\Rightarrow \mathrm{rank}(A) = n = 3 \Rightarrow \det(A) \neq 0$.

Task 7

Given are the functions:

$$f(x) = 5x^3 - \frac{15}{2}x^2 - 30x + 50$$

$$g(x) = 2 - \frac{1}{4}x^3$$

$$h(x) = 1 - 2x^3$$

(a) Determine the limits values

$$\lim_{x\to\infty}\frac{f(x)}{g(x)} = \frac{5x^3 - 7.5x^2 - 30x + 50}{2 - 0.25x^3}$$

Deg(top)=Deg(bottom):

$$\lim_{x\to\infty} f(x) = \frac{leading\ coefficient\ of\ top}{leading\ coefficient\ of\ bottom} = -\frac{5}{0.25} = -20$$

$$\lim_{x\to 0}\frac{f(x)}{g(x)} = \frac{5x^3 - 7.5x^2 - 30x + 50}{2 - 0.25x^3} = \frac{50}{2} = 25$$

$$\lim_{x\to 2}\frac{f(x)}{g(x)} = \frac{5x^3 - 7.5x^2 - 30x + 50}{2 - 0.25x^3}$$

$$f(2) = 5\cdot 8 - 7.5\cdot 4 - 30\cdot 2 + 50 = 40 - 30 - 60 + 50 = 0$$

$$g(2) = 2 - 0.25\cdot 8 = 2 - 2 = 0$$

$\lim_{x\to 2}\frac{f(x)}{g(x)}$ has the form $\frac{0}{0} \rightarrow$ L'Hôpital Rule

$$\lim_{x\to 2}\frac{f(x)}{g(x)} = \frac{5x^3 - 7.5x^2 - 30x + 50}{2 - 0.25x^3} = \frac{15x^2 - 15x - 30}{2 - 0.75x^2} = \frac{0}{-1} = 0$$

(b) Determine the positions of the local extrema of f : where does this function reach a minimum, where a maximum?

$$f(x) = 5x^3 - \frac{15}{2}x^2 - 30x + 50$$

$$f'(x) = 15x^2 - 15x - 30$$

Critical points:

$$15x^2 - 15x - 30 = 0$$

$$x_{1,2} = \frac{15 \pm \sqrt{15^2 - 4 \cdot 15 \cdot (-30)}}{-2 \cdot 15} = \frac{15 \pm 45}{-30}$$

$$x_1 = -2;\ x_2 = 1$$

$$f''(x) = 30x - 15$$

$$x_1 = -2;\ f''(-2) = 30 \cdot (-2) - 15 = -75 < 0 \ \rightarrow local\ maximum$$

$$x_2 = 1;\ f''(1) = 30 \cdot 1 - 15 = 15 < 0 \ \rightarrow local\ minimum$$

(c) Draw the function h : flipped parabola, with top point at (0; 1).

- Prove that h is not injective.

"h injective" would mean: for all a, b: $a \neq b \Rightarrow h(a) \neq h(b)$.

But we have (e.g.): $h(-1) = 1 - 2\cdot(-1)^2 = -1 = 1 - 2\cdot 1^2 = h(1)$.

So h cannot be injective.

(d) Calculate $h(g(x))$ Simplify the term as far as possible

$$h(g(x)) = 1 - 2\left(2 - \frac{1}{4}x^3\right)^2 = 1 - 2\left(4 - x^3 + \frac{1}{16}x^6\right) =$$

$$-7 + 2x^3 - \frac{1}{8}x^6$$

Task 8. Extremal points of functions of two variables

Given is the function
$f(x,y) = 4x^2y + 2xy - 3y^2 + 5$

(a)Calculate the following partial derivatives:

$f_x = 8xy + 2y$
$f_y = 4x^2 + 2x - 6y$
$f_{xx} = 8y$
$f_{xy} = 8x + 2$
$f_{yy} = -6$

(b)Calculate all critical points (x,y) of f (i.e., all points where f_x and f_y are both 0)

$$\begin{cases} 8xy + 2y = 0 \\ 4x^2 + 2x - 6y = 0 \end{cases}$$

$$2y(4x+1) = 0 \rightarrow y = 0; x = -\frac{1}{4}$$

$$y = 0 \quad x = 0 \ or\ x = -\frac{1}{2}$$

$$x = -\frac{1}{4}\ y = -\frac{1}{24}$$

(c) Indicate for each critical point if it is a saddle point or a local extremal point of f, and in the latter case, if it is a maximum or a minimum

$$H(f) = \begin{pmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{pmatrix} = \begin{pmatrix} 8y & 8x+2 \\ 8x+2 & -6 \end{pmatrix}$$

$\boldsymbol{y = 0;\ x = 0}$

$$D = f_{xx}(x_0,y_0)\cdot f_{yy}(x_0,y_0) - f_{xy}(x_0,y_0)\cdot f_{yx}(x_0,y_0) =$$

$$0\cdot(-6) - 2\cdot 2 = -4 < 0 \rightarrow saddle\ point$$

$$\boldsymbol{x = -\frac{1}{2}; y = 0}$$

$$D = f_{xx}(x_0,y_0)\cdot f_{yy}(x_0,y_0) - f_{xy}(x_0,y_0)\cdot f_{yx}(x_0,y_0) =$$

$$0\cdot(-6) - (-2)\cdot(-2) = -4 < 0 \rightarrow saddle\ point$$

$$x = -\frac{1}{4}; y = -\frac{1}{24}$$

$$D = f_{xx}(x_0, y_0) \cdot f_{yy}(x_0, y_0) - f_{xy}(x_0, y_0) \cdot f_{yx}(x_0, y_0) =$$

$$8\left(-\frac{1}{24}\right) \cdot (-6) - \left(8\left(-\frac{1}{4}\right) + 2\right)^2 = 2 > 0$$

$$f_{xx}(x_0, y_0) = -\frac{1}{3} < 0 \rightarrow local\ maximum$$

Task 9 Integration

Calculate the values of the following integrals:

(a)

$$\int_0^2 (x^3 - x)\,dx = \frac{1}{4}x^4 - \frac{1}{2}x^2\Big|_0^2 = \frac{1}{4}16 - \frac{1}{2}4 - 0 = 2$$

(b)

$$\int_0^{\pi} \sin x\,dx = -\cos x\Big|_0^{\pi} = -\cos\pi - (-\cos 0) = 1 + 1 = 2$$

Task 10

(a)

A B C D
A B D C
A C B D
A C D B
A D B C
A D C B
B A C D
B A D C
B C A D
B C D A
B D A C
B D C A
C A B D
C A D B
C B A D
C B D A
C D A B
C D B A
D A B C
D A C B
D B A C
D B C A
D C A B
D C B A

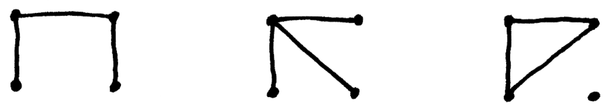
$(P(4) = 24)$

(b) Each of the n letters can occur in the first position.
The other $(n-1)$ letters can occur in $P(n-1)$ permutations on the rest of the positions.
Together, there are $n \cdot P(n-1)$ possibilities to permute all n letters.

$\Rightarrow \quad P(n) = n \cdot P(n-1)$.

(From this, it follows: $P(n) = n!$.)

Task 11



2 are connected

1 is unconnected

Task 12

(a) $8 \cdot 10^6 \cdot 2 \text{ bit} = 2 \cdot 10^6 \text{ Byte} = 2 \text{ MB}$

(b) $\underset{\substack{\uparrow \\ 1024}}{2^{10}} \cdot 2^{10} \cdot 24 \text{ bit} = 2^{20} \cdot 2^3 \cdot 3 \text{ bit} = 3 \cdot 2^{20} \text{ Byte} \approx 3 \cdot 10^6 \text{ Byte} = 3 \text{ MB}$

$\Rightarrow$ (b) needs more storage capacity.

Task 13

(a) x + Math.sqrt(1 - x*x)/2 $\longrightarrow$ $x + \frac{\sqrt{1-x^2}}{2}$

(b) possible runtime error: root of negative number.
Condition to be checked to avoid this: $1-x^2 \geq 0$
(or, equivalently: $x^2 \leq 1$, or: $|x| \leq 1$.)

Task 14

Bud ==> Shoot [RU(45) Bud] [RU(-45) Bud] ;

after a second application of this rule we get:

Shoot [RU(45) Shoot [RU(45) Bud][RU(-45) Bud]]
[RU(-45) Shoot [RU(45) Bud][RU(-45) Bud]]

With initially vertical direction of Shoot (i.e., F), the geometrical interpretation looks like this: