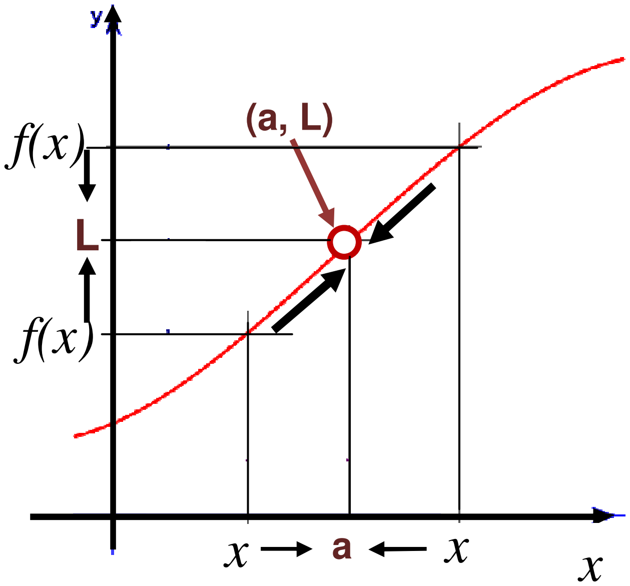
Limits of Functions



Informal Definition:

If the values of $f(x)$ can be made as close to L as we like by taking values of x sufficiently close to a [but not equal to a] **then** we write

$$\lim_{x \to a} f(x) = L$$

or

$$f(x) \to L \text{ as } x \to a$$

Observe:

- "$x \to a$" means x can approach a **from either side**
- On a sketch, the graph of $f(x)$ approaches the 2-D plane location [destination] called (a, L), but the graph itself may have no point $(a, f(a))$ occupying that location!

L may not be $f(a)$

The language to describe how the outputs $f(x)$ behave as the inputs x approaches a number L

$$\lim_{x \to a} f(x) = L$$

Example:

$$\lim_{x\to 0}\left[\frac{x}{\sqrt{x+1}-1}\right] -?$$

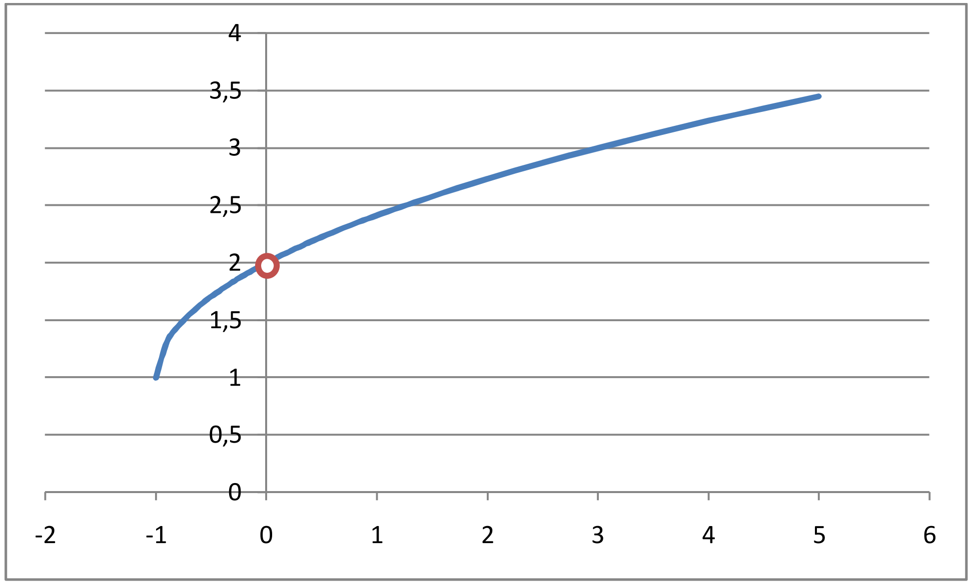
We examine the graph

Domain of $f(x)$

$$\sqrt{x+1}-1 \neq 0, x \neq 0$$

$$x+1 \geq 0, x \geq -1$$

$$\{x \in \mathbb{R} | x \geq -1, x \neq 0\}$$



Conjecture:

$$\lim_{x \to 0} \left[\frac{x}{\sqrt{x+1} - 1} \right] = 2$$

General definition:

Let $f(x)$ be a function and a a real number (that may be or may be not in the **domain of $\boldsymbol{f}$**). We say that the limit as x approaches a of $f(x)$ is L, written

$$\lim_{x \to a} f(x) = L$$

if $f(x)$ can be made arbitrarily close to L by choosing x sufficiently close to (but not equal to) a.

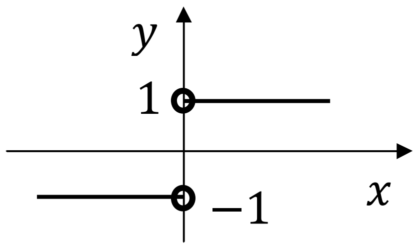
If no such number exists, then we say that

$\lim_{x \to a} f(x)$ does not exist .

Warning: Not all limits exist!

Example:

$$f(x) = \frac{|x|}{x} = \begin{cases} 1 \ if \ x > 0 \\ -1 \ if \ x < 0 \end{cases}$$



- $x \to 0$ from the left, $f(x) \to -1$
- $x \to 0$ from the right, $f(x) \to 1$

So $\lim_{x \to 0} f(x)$ has no meaning!

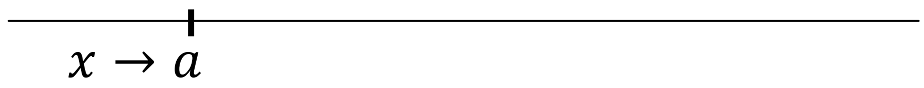
Two-Sided and One-Sided Limits

Notation

"x approaches a from the left"

$x \to a^-$ [minus in a superscript position] or $x \uparrow a$ [comes up to a] or $x \nearrow a$

$$\lim_{x \to a^-} f(x) = L$$



"x approaches a from the right"

$x \to a^+$ [plus in a superscript position] or $x \downarrow a$ [comes down to a] or $x \searrow a$

$$\lim_{x \to a^+} f(x) = L$$

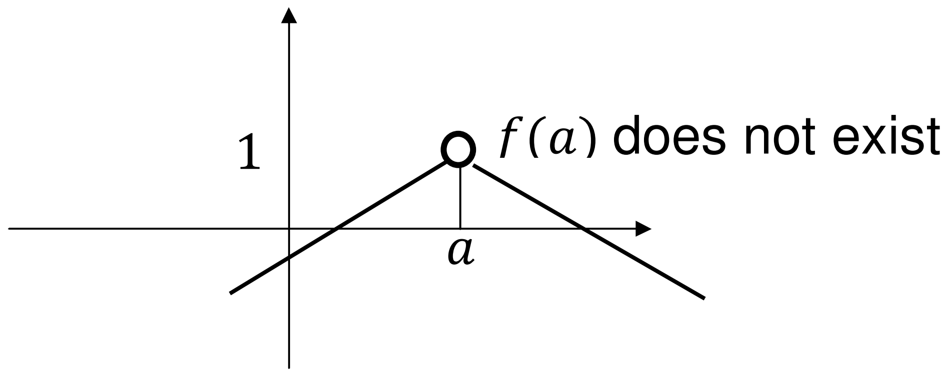
Relationship between Two-Sided and One-Side Limits: Theorem

$$\lim_{\underbrace{x \to a}_{\text{two-sided}}} f(x) = L$$

$\Longleftrightarrow$ [if and only if]:

- $\lim_{x \to a^-} f(x) = L$ exists
- $\lim_{x \to a^+} f(x) = L$ exists
- and both equal L

Example:



$$\lim_{x\to a^-} f(x) = 1 = \lim_{x\to a^+} f(x)$$

The Algebra of Limits as $x \to a$

Basic Limits as $x \to a, a^+, a^-$ used in polynomial functions and rational functions.

The constant function

$$\lim_{x \to a} (k) = k$$

The identity function: $f(x)$

$$\lim_{x \to a} (x) = a$$

The reciprocal ("flip over") function: $f(x) = \frac{1}{x}$

$$\lim_{x \to 0^-} \left(\frac{1}{x}\right) = -\infty$$

$$\lim_{x \to 0^+} \left(\frac{1}{x}\right) = \infty$$

Limits of Sums, Differences, Products, Quotients and Roots

The “Rules” of Algebra for Limits

Let a be any real number and

$$\lim_{x \to a} f(x) = L_1$$

$$\lim_{x \to a} g(x) = L_2$$

then

$$\lim_{x \to a}[f(x) + g(x)] = \lim_{x \to a} f(x) + \lim_{x \to a} g(x) = L_1 + L_2$$

$$\lim_{x \to a}[f(x) - g(x)] = \lim_{x \to a} f(x) - \lim_{x \to a} g(x) = L_1 - L_2$$

$$\lim_{x \to a}[f(x) \cdot g(x)] = \lim_{x \to a} f(x) \cdot \lim_{x \to a} g(x) = L_1 \cdot L_2$$

$$\lim_{x \to a}|f(x)| = \left|\lim_{x \to a} f(x)\right| = |L_1|$$

$$\lim_{x \to a}[kf(x)] = k \cdot \lim_{x \to a} f(x) = k \cdot L_1$$

$$\lim_{x\to a}[f(x)^n] = \left[\lim_{x\to a} f(x)\right]^n = \mathrm{L}_1^n$$

$$\lim_{x\to a}\left[\frac{f(x)}{g(x)}\right] = \frac{\lim_{x\to a} f(x)}{\lim_{x\to a} g(x)} = \frac{L_1}{L_2}$$

Provided $L_2 \neq 0$

$$\lim_{x\to a} \sqrt[n]{f(x)} = \sqrt[n]{\lim_{x\to a} f(x)} = \sqrt[n]{L_1}$$

Provided when n= even then $L_1 \geq 0$

Limits of Polynomial Function

Polynomial Expressions

A monomial (one-term polynomial) has the form

$$a_n \cdot x^n$$

A real number constant called a "coefficient"
Subscript n is a label

x- a variable

n=0,1,2, 3,…
not negative
called the **degree** of the monomial

Two monomial with the same degree and same variable are called "like terms":
$a_n \cdot x^n$; $b_n \cdot x^n$ - "like terms"

A polynomial in one variable has the standard form: [higher powers → lower powers]

$$a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0$$

$a_n \neq 0$ leading coefficient

By the “Rules of Algebra” for Limits we can break down **polynomials** into simpler parts

Example:

$$\lim_{x\to a}[f(x) \pm g(x)] = \lim_{x\to a} f(x) \pm \lim_{x\to a} g(x)$$

$$\lim_{x\to a}[f(x)^n] = \left[\lim_{x\to a} f(x)\right]^n$$

$$\lim_{x\to a}(k) = k$$

$$\lim_{x\to a}(x) = a$$

$$\lim_{x\to 5}[x^2 - 4x + 3] = \lim_{x\to 5}[x^2] - \lim_{x\to 5}[4x] + \lim_{x\to 5}[3] = \left(\lim_{x\to a}[x]\right)^2 - 4\lim_{x\to a}[x] + 3 =$$

$$\lim_{x\to a}[kf(x)] = k \cdot \lim_{x\to a} f(x)$$

$$= 5^2 - 4 \cdot 5 + 3 = 8$$

For any polynomial function

$$\lim_{x \to a} p(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0 = p(a)$$

For polynomial, this limit is the same as “substitution of a for x”

Limits of Rational Functions $\frac{p(x)}{q(x)}$ and the appearance of $\frac{\mathbf{0}}{\mathbf{0}}$

There are 3 cases to consider

Case 1: $q(a) \neq 0$ Limit $= \frac{p(a)}{q(a)}$

Example:

$$\lim_{x \to 2} \left[\frac{5x^3 + 4}{x - 3} \right] \begin{matrix} \leftarrow p(x) \\ \leftarrow q(x) \end{matrix} \quad a = 2$$

$$= \frac{\lim_{x \to 2} [5x^3 + 4]}{\lim_{x \to 2} [x - 3]} = \frac{\overbrace{5 \cdot 2^3 + 4}^{p(a)}}{\underbrace{2 - 3}_{q(a) \neq 0}} = -44$$

Case 2: $p(a) \neq 0 \; and \; q(a) = 0$ Limit does not exist (division by 0!)

$$f(x) = \frac{\overbrace{1}^{p(x)}}{\underbrace{x-a}_{q(x)}}$$

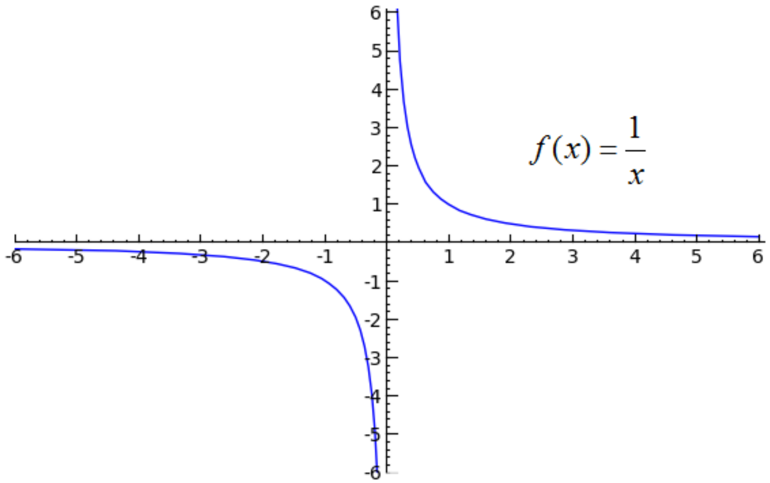
$$\lim_{x \to a^-} \left[\frac{1}{x-a}\right] = -\infty; \; \lim_{x \to a^+} \left[\frac{1}{x-a}\right] = +\infty$$

Classic Examples:

$$f(x) = \frac{1}{x}$$

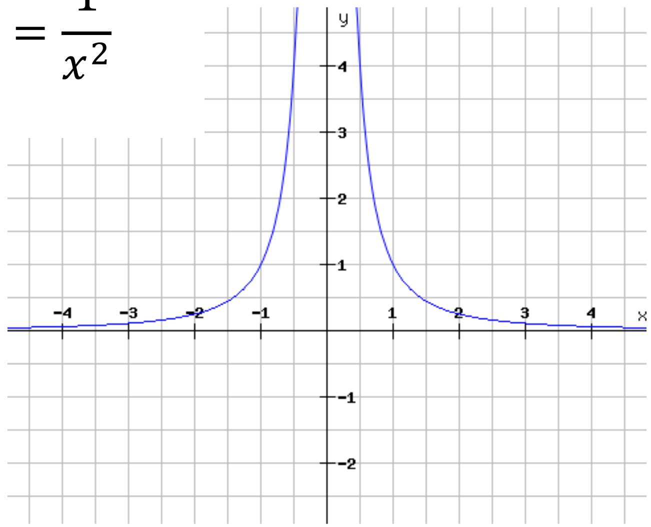
$$\lim_{x \to 0^-} \left[\frac{1}{x}\right] = -\infty$$

$$\lim_{x \to 0^+} \left[\frac{1}{x}\right] = +\infty$$



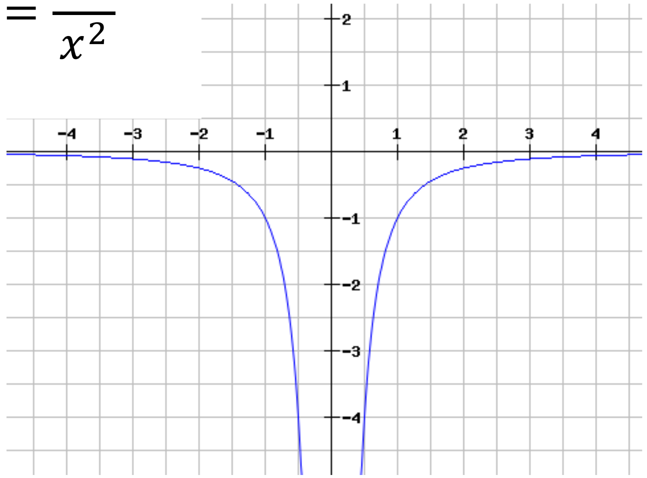
$$\lim_{x \to a}\left[\frac{1}{(x-a)^2}\right] = \infty$$

$$f(x) = \frac{1}{x^2}$$



$$\lim_{x \to a}\left[\frac{-1}{(x-a)^2}\right] = -\infty$$

$$f(x) = \frac{-1}{x^2}$$



Case 3: $p(a) = 0 \; and \; q(a) = 0$ Limit $\frac{p(a)}{q(a)} = \frac{0}{0}$ an indeterminate form: We cannot determine whether the limit exists or not, without more work!

Example:

$$\lim_{x \to 2} \left[\frac{x^2 - 4}{x - 2} \right] \begin{bmatrix} \leftarrow p(x) \\ \leftarrow q(x) \end{bmatrix} both \; p(2) = 0 = q(2)$$

$Factor + cancel$

$$\lim_{x \to 2} \left[\frac{(x-2)(x+2)}{x - 2} \right] = \lim_{x \to 2} [(x + 2)] = 4$$

This is only one particularly technique! Does not work always!

The Algebra of Limits as $x \to \pm\infty$: End Behavior

Basic Limits:

The constant function

$$\lim_{x \to -\infty} (k) = k$$

and

$$\lim_{x \to \infty} (k) = k$$

The identity function: $f(x)$

$$\lim_{x \to -\infty} (x) = -\infty$$

$$\lim_{x \to \infty} (x) = \infty$$

The reciprocal ("flip over") function: $f(x) = \frac{1}{x}$

$$\lim_{x \to -\infty} \left(\frac{1}{x}\right) = 0$$

$$\lim_{x \to \infty} \left(\frac{1}{x}\right) = 0$$

Limits of Sums, Differences, Products, Quotients and Roots

The "Rules" of Algebra for Limits applied to $x \to -\infty$ or $x \to \infty$

We only state for $x \to \infty$ case

As before, suppose:

$$\lim_{x \to \infty} f(x) = L_1$$

$$\lim_{x \to \infty} g(x) = L_2$$

then

$$\lim_{x \to \infty} [f(x) + g(x)] = \lim_{x \to \infty} f(x) + \lim_{x \to \infty} g(x) = L_1 + L_2$$

$$\lim_{x \to \infty} [f(x) - g(x)] = \lim_{x \to \infty} f(x) - \lim_{x \to \infty} g(x) = L_1 - L_2$$

$$\lim_{x \to \infty} [f(x) \cdot g(x)] = \lim_{x \to \infty} f(x) \cdot \lim_{x \to \infty} g(x) = L_1 \cdot L_2$$

$$\lim_{x \to \infty} |f(x)| = \left| \lim_{x \to \infty} f(x) \right| = |L_1|$$

$$\lim_{x\to\infty}[kf(x)] = k \cdot \lim_{x\to\infty} f(x) = k \cdot L_1$$

$$\lim_{x\to\infty}[f(x)^n] = \left[\lim_{x\to\infty} f(x)\right]^n = \mathrm{L}_1^n$$

$$\lim_{x\to\infty}\left[\left(\frac{1}{x}\right)^n\right] = \left[\lim_{x\to\infty}\frac{1}{x}\right]^n = 0$$

$$\lim_{x\to\infty}\left[\frac{f(x)}{g(x)}\right] = \frac{\lim_{x\to\infty} f(x)}{\lim_{x\to\infty} g(x)} = \frac{L_1}{L_2}$$

Provided $L_2 \neq 0$

$$\lim_{x\to\infty}\sqrt[n]{f(x)} = \sqrt[n]{\lim_{x\to\infty} f(x)} = \sqrt[n]{L_1}$$

Provided when n= even then $L_1 \geq 0$

Limits of Polynomial Functions: Two End Behaviors

A polynomial function

$$f(x) = c_n x^n + c_{n-1} x^{n-1} + \cdots + c_1 x + c_0$$

Where $c_n \neq 0$

The "two end behaviors" are that as $x \to \infty$ (the rightward end) or $x \to -\infty$ (the leftward end)

Then

$$\left.\begin{matrix} f(x) \to \infty \\ f(x) \to -\infty \end{matrix}\right\} \text{ The two possibilities}$$

Observe:

$$f(x) = c_n x^n + c_{n-1} x^{n-1} + c_1 x + c_0 = x^n \left[c_n + \underbrace{\frac{c_{n-1}}{x} + \cdots + \frac{c_1}{x^{n-1}} + \frac{c_0}{x^n}}_{go\ to\ 0} \right]$$

So, the "end behavior" of $f(x)$ matches the "end behavior" of $c_n x^n$

Theorem:

$$\lim_{x\to\pm\infty}[c_n x^n + c_{n-1}x^{n-1} + c_1 x + c_0] = \lim_{x\to\pm\infty}[c_n x^n]$$

Example:

$$\lim_{x\to-\infty}[-4x^8 + 17x^5 + 3x^4 + 2x - 50] = \lim_{x\to-\infty}[-4x^8] = -\infty$$

Limits of Rational Functions: Three Types of End Behavior

$$f(x) = \frac{p(x)}{q(x)} \begin{array}{l} \leftarrow top \\ \leftarrow botton \end{array}$$

Remember: The Degree of a polynomial is the exponent of the highest power of x in the polynomial

Type 1. Deg(top)=Deg(bottom)

$$\lim_{x\to\infty} f(x) = \frac{\text{leading coefficient of top}}{\text{leading coefficient of bottom}}$$

Example:

$$f(x) = \frac{-x}{7x+4}$$

$$\lim_{x\to\infty} f(x) = \lim_{x\to\infty}\left[\frac{-x}{7x+4}\right] = \lim_{x\to\infty}\left[\frac{-1}{7+\underbrace{\frac{4}{x}}_{\searrow 0}}\right] = -\frac{1}{7} = \frac{l.c.\,of\ top}{l.c.\,of\ bottom}$$

Type 2. Deg(top)<Deg(bottom)

$$\lim_{x\to\infty} f(x) = 0$$

Example:

$$f(x) = \frac{5x+2}{2x^3-1}$$

$$\lim_{x\to\infty} f(x) = \lim_{x\to\infty}\left[\frac{5x+2}{2x^3-1}\right] = \lim_{x\to\infty}\left[\frac{\frac{5}{x^2}+\frac{2}{x^3}}{2-\frac{1}{x^3}}\right] = 0$$

Always zero

$y = 0$ the (x-axis) is a horizontal asymptote

Type 3. Deg(top)>Deg(bottom)

If $leading\ coefficient\ of\ top > 0$

$$\left.\begin{matrix} \infty & if & x \to \infty \\ -\infty & if & x \to -\infty \end{matrix}\right\} \text{ Always one of these}$$

If $leading\ coefficient\ of\ top < 0$

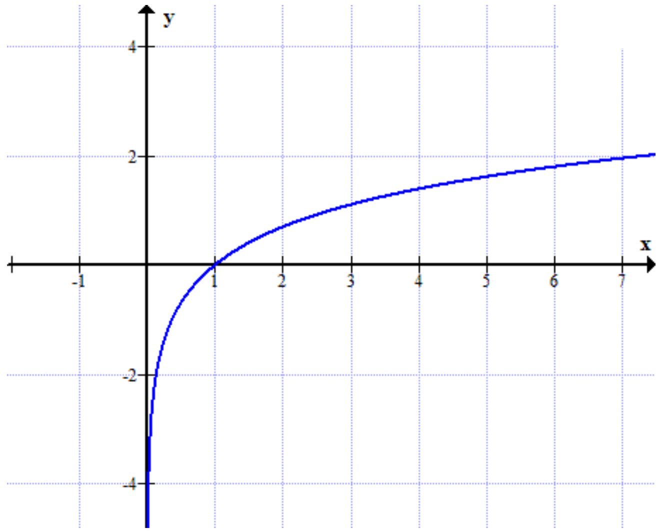
$$\left.\begin{matrix} \infty & if & x \to -\infty \\ -\infty & if & x \to \infty \end{matrix}\right\} \text{ Always one of these}$$

Example:

$$f(x) = \frac{x^2 + 4x + 5}{x - 1}$$

$$\lim_{x\to\infty} f(x) = \lim_{x\to\infty}\left[\frac{x^2 + 4x + 5}{x - 1}\right] = \lim_{x\to\infty}\left[\frac{1 + \frac{4}{x} + \frac{5}{x^2}}{\frac{1}{x} - \frac{1}{x^2}}\right] = \infty$$

Limits of $\ln(x)$

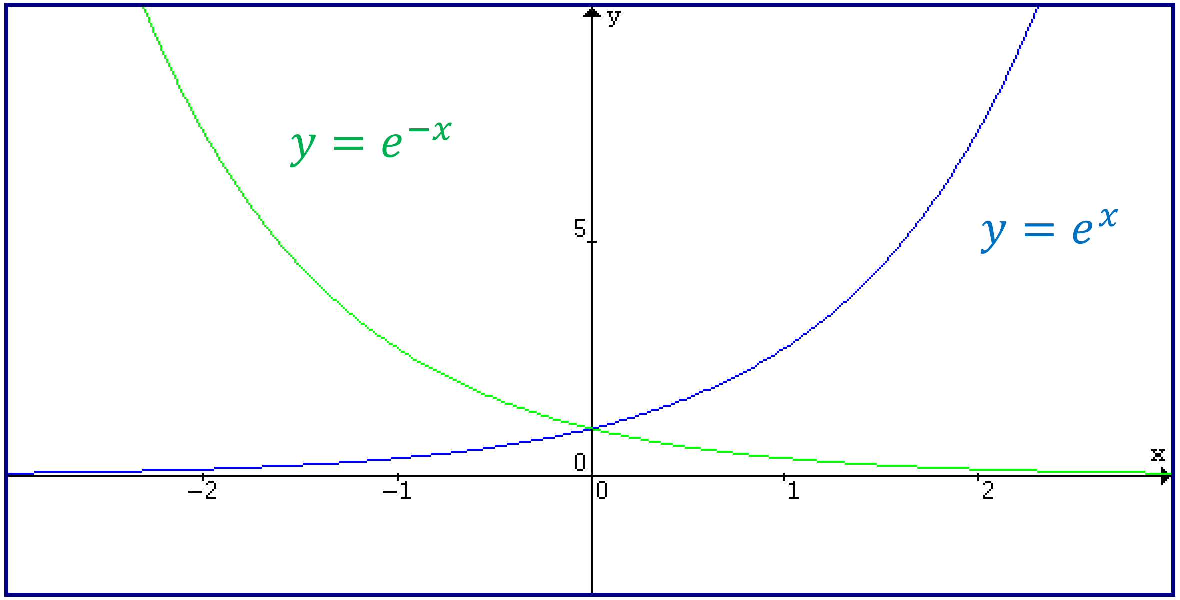


$$\lim_{x \to \infty} [lnx] = \infty$$

note, that $\lim_{x \to -\infty} lnx$ makes no sense

$$\lim_{x \to 0^+} [lnx] = -\infty$$

Limits of e^x



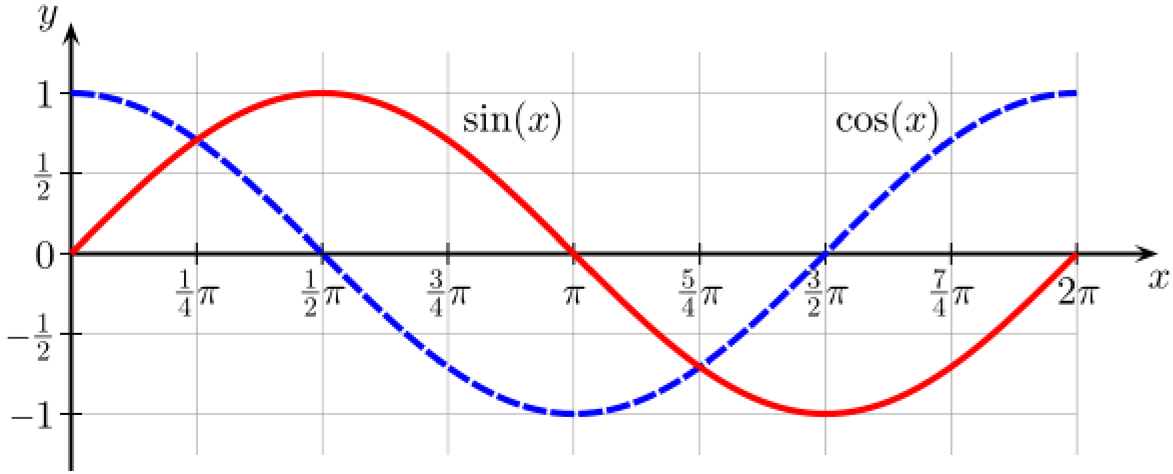
$$\lim_{x \to \infty} [e^x] = \infty$$

$$\lim_{x \to -\infty} [e^x] = 0$$

$$\lim_{x \to \infty} [e^{-x}] = 0$$

$$\lim_{x \to -\infty} [e^{-x}] = -\infty$$

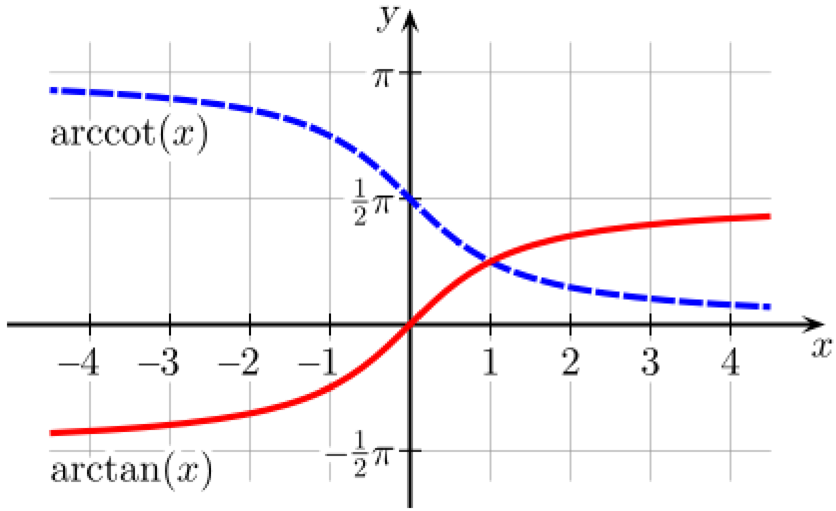
Limits of Trigonometric Functions



$$\lim_{x \to 0} cosx = cos0 = 1$$

$$\lim_{x \to 0} sinx = sin0 = 0$$

Limits of $arctanx$ and $arccotx$



$$\lim_{x\to\infty} arctanx = \frac{\pi}{2}$$

$$\lim_{x\to-\infty} arctanx = -\frac{\pi}{2}$$

$$\lim_{x\to\infty} arccotx = 0$$

$$\lim_{x\to-\infty} arccotx = \pi$$