Computer Sciences and Mathematics:

Part 2: Basics of Calculus

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Mathematics: Topics for the next 4 weeks

1. Sequences, Partial sums, Series
2. Limit of a Function
3. Differentiation
4. Partial Derivatives
5. Curve Discussion and Extreme Value Problem
6. Integration

Sources:

Lectures with Professor Richard Delaware,

University of Missouri - Kansas City (USA)**:**

http://www.youtube.com/playlist?list=PLF5E22224459D23D9

http://www.youtube.com/playlist?list=PLDE28CF08BD313B2A

Lectures with Professor Gregory L. Naber,

Drexel University (USA):

http://www.gregnaber.com/lectures/calculus/

Wikipedia

In biosciences the mathematical models are largely used to describe properties of systems of interest.

To understand the principles of the modeling one needs elementary basics of mathematics.

Mathematics provides a tool for models in biosciences: Language, rules, techniques, algorithms.

The main teaching objective of the course: To become acquainted with the basics of **calculus**, that includes the theories of differentiation, integration, measure, limits, infinite series, and analytic functions.

Sequences

Consider the infinite "list" of terms:

formula for n^{th} term

$$\frac{1}{1}, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \dots, \frac{1}{n}$$

In general:

$$a_1, a_2, a_3, \dots a_n, \qquad a_n - \ n^{th} term$$

or as a function

$$a(1), a(2), a(3), a(4), \dots, a(n)$$

A **sequence** is an ordered set of numbers that most often follows some rule (or pattern) to determine the next term in the order.

Definition:

- A sequence is a **function** whose Domain is $\mathbb{N} = \{1, 2, 3, 4, \ldots, n\}$

- In practice, we usually refer to the infinite list of its Range values as the sequence:

$$a(1), a(2), a(3), a(4), \ldots, a(n)$$

Notation:

$$\{a_n\} = \{a_1, a_2, a_3, \ldots a_n\}$$

or

$$\{a_n\}_{n=1}^{\infty} = \{a_1, a_2, a_3, \ldots a_n\}$$

A sequence is often given by the n^{th} term formula (also called **general term**)

Exercise: Write the first 5 terms:

$$\{a_n\} = \frac{1}{n}, n = 1,2,3,4, \ldots$$

Solution:

$$\frac{1}{1}, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}$$

Exercise: Write the first 5 terms:

$$a_n = \frac{1}{2^n}, n = 1,2,3,4, \ldots$$

Solution:

$$\frac{1}{2^1}, \frac{1}{2^2}, \frac{1}{2^3}, \frac{1}{2^4}, \frac{1}{2^5}$$

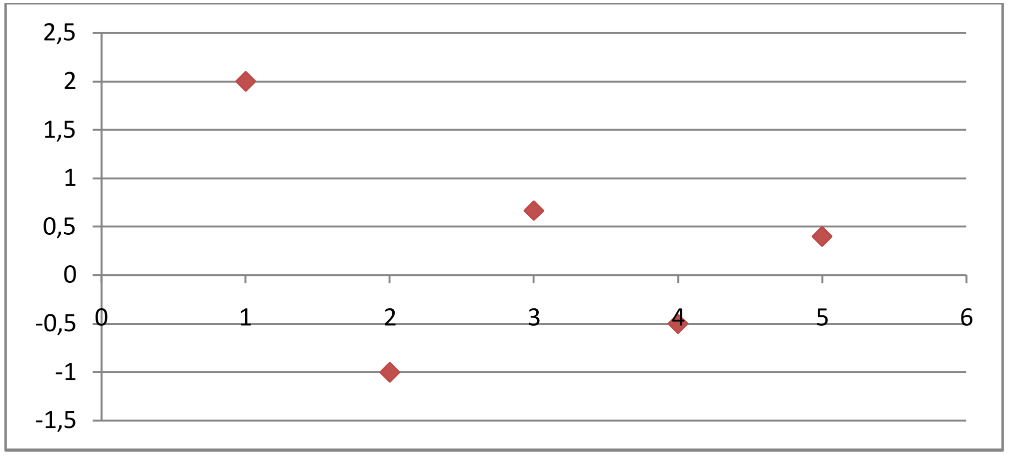
Exercise 3: Write the first 5 terms:

$$\{b_n\} = \left\{(-1)^{n-1} \cdot \left(\frac{2}{n}\right)\right\}, n = 1,2,3,4, \ldots$$

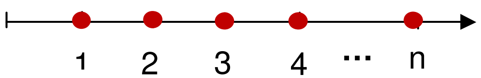
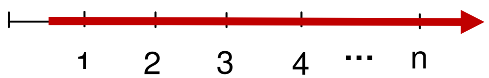
Solution 3:

$$\{b_n\} = \left\{\frac{2}{1}, \frac{-2}{2}, \frac{2}{3}, \frac{-2}{4}, \frac{2}{5}\right\}$$

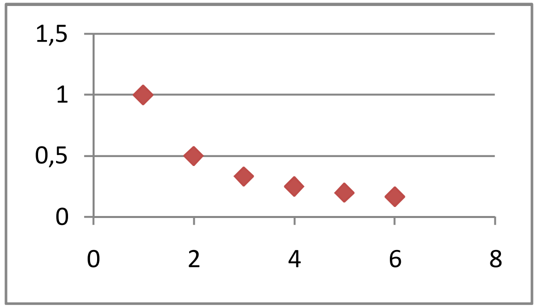
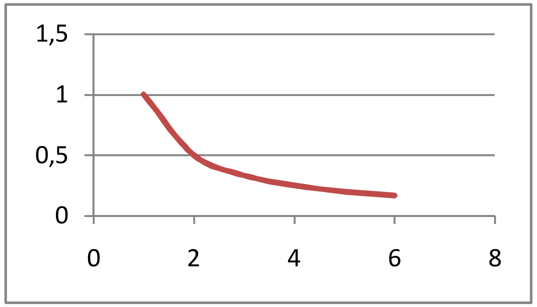
Graph 3:



Distinctions between sequences and functions:

Sequence	**Function**
$a(n) = a_n, n \in \mathbb{N}$	$f(x), x \in \mathbb{R}$
Dom a: Discrete Numbers: 1, 2, 3…	Dom f: Continuous Intervals
	
Range $a(n)$ $a = \{a_1, a_2, a_3, \dots a_n, \dots \dots\}$	Range $f(x)$= some Interval

Example:

Sequence	Function
$a_n = \frac{1}{n}, n \in \mathbb{N}$	$f(x) = \frac{1}{x}, x \in [1, \infty)$
$Ran\ a = \left\{1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots\right\}$	$Ran\ f(x) = (0,1]$
	
Graph: Consisting of discrete discontinuous <u>Dots</u>	Graph: Continuous Curve

Warning: You cannot determine a sequence from only a finite number of terms!

Example:

a_1	a_2	a_3
1	3	9
1	3	9
1	3	9

a_1	a_2	a_3	a_4	…	a_n
1	3	9	27	…	3^{n-1}
1	3	9	19	…	$1 + 2(n-1)^2$
1	3	9	11	…	$8n + \frac{12}{n} - 19$

Often a sequence is given by a recursive formula

- Stating its 1st term (s), then
- Writing a formula for the n^{th} term involving some preceding terms. This is called **a recursive formula**

Example:

$$a_1 = 1$$

$$\underbrace{a_n}_{subsequent} = 4 \cdot \underbrace{a_{n-1}}_{previous} \; : recursive\ formula$$

Solution:

$$a_1 = 1$$

$$a_2 = 4 \cdot a_1 = 4 \cdot 1 = 4$$

$$a_3 = 4 \cdot a_2 = 4 \cdot 4 = 16$$

$$a_4 = 4 \cdot a_3 = 4 \cdot 16 = 64$$

Limits of sequences

Some sequences "approach" a number as you move out of the sequence: e.g. the sequence

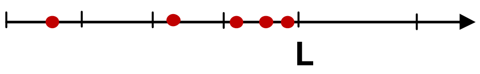
$$1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots$$

"approaches" 0

A Sequence

$$\{a_n\}_{n=1}^{\infty} = \{a_1, a_2, a_3, \dots\}$$

is said to **converge** to a number L (called a **limit of the sequence**) if any interval around L (however small) contains all the terms of a sequence beyond some point



In this case we write:

$$L = \lim_{n\to\infty} a_n$$

If no such number exists we say that the sequence **diverges.**

Example 1

$$\{3^n\}_{n=1}^{\infty} = \{3, 3^2, 3^3, \dots\}$$

clearly diverges.

Example 2

$$\left\{\left(\frac{1}{2}\right)^{n}\right\}_{n=1}^{\infty} = \left\{\frac{1}{2}, \left(\frac{1}{2}\right)^{2}, \left(\frac{1}{2}\right)^{3}, \ldots\right\}$$

converges to 0.

Example 3

$$\{(1)^n\}_{n=1}^{\infty} = \{1, 1, 1, \ldots\}$$

converges to 1

In general:

$$\{x^n\}_{n=1}^{\infty}$$

converges to 0, if $-1 < x < 1$, converges to 1 if $x = 1$ and diverges for every other value of x

Shorthand: Summation notation.

The sum of n terms can be written as

Where k ends

$$a_1 + a_2 + a_3 + \cdots + a_n = \sum_{k=1}^{n} a_k$$

The form (formula) for the k^{th} term of our sequence

„Sigma“- greek „s“

[stands for: sum]

Index, starts at $k = 1$

Examples

1.

$$\sum_{k=1}^{n}\left(\frac{1}{k}\right)=\frac{1}{1}+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{n}$$

2.

$$1^2+2^2+3^2+\cdots+n^2=\sum_{k=1}^{n}(k)^2$$

3.

$$\sum_{k=5}^{8}\left[(-1)^{k+1}\cdot 2^k\right]=2^5-2^6+2^7-2^8$$

Note: The index k must not begin at 1 or end at n

Other letters can be used: i, j

Example:

$$\sum_{i=2}^{5} i = 2 + 3 + 4 + 5$$

placeholder

$$\sum_{\square=2}^{5} \square$$

Theorem: Properties of Sequences

If $\{a_n\}$, $\{b_n\}$ are sequences, and $c \in \mathbb{N}$

1. $\sum_{k=1}^{n}(c \cdot a_k) = c \cdot \sum_{k=1}^{n}(a_k)$

2+3. $\sum_{k=1}^{n}(a_k \pm b_k) = \sum_{k=1}^{n} a_k \pm \sum_{k=1}^{n} b_k$

4. $\sum_{k=1}^{n} a_k = \sum_{k=1}^{j} a_k + \sum_{k=j+1}^{n} a_k$, where $1 < j < n$ breaks into 2 pieces

5. $\sum_{k=1}^{n} c = n \cdot c$

Examples:

1.

$$\sum_{k=3}^{8} 9 = 9 + 9 + 9 + 9 + 9 + 9 = 6 \cdot 9$$

2.

$$\sum_{k=1}^{3} 5 \cdot \frac{1}{k} = 5\frac{1}{1} + 5\frac{1}{2} + 5\frac{1}{3} = 5\left(\frac{1}{1} + \frac{1}{2} + \frac{1}{3}\right)$$

Example:

Given are the following measurements for y_{ij}:

j \ i	**1**	**2**	**3**	**4**	**5**
1	1	2	1	3	6
2	3	5	3	1	5
3	4	3	2	5	1
4	6	8	2	3	2

Compute the following sums:

$$\sum_{i=3}^{5} \sum_{j=2}^{4} y_{ij}$$

Solution:

j \ i	**1**	**2**	**3**	**4**	**5**
1	1	2	1	3	6
2	3	5	3	1	5
3	4	3	2	5	1
4	6	8	2	3	2

$$\sum_{i=3}^{5}\sum_{j=2}^{4} y_{ij} = \sum_{i=3}^{5}(y_{i2}+y_{i3}+y_{i4}) = y_{32}+y_{33}+y_{34}+y_{42}+y_{43}+y_{44}+y_{52}+y_{53}+y_{54}$$

$$= 3+2+2+1+5+3+5+1+2 = 24$$

Partial Sums

Given an infinite sequence $\{a_k\} = \{a_1,\ a_2,\ a_3, \ldots\}$

the sum of the first n terms is

$$a_1 + a_2 + a_3 + \cdots a_n$$

the $\boldsymbol{n^{th}}$ **partial sum**

Notation:

$$S_n = \sum_{k=1}^{n} a_k, \quad n = 0,1,2, \ldots$$

A partial sum is a sum of part of the sequence

Arithmetic Sequence

Definition: Let $a, d \in \mathbb{R}$. An arithmetic sequence has the standard form

$$\{a, a+d, a+2d, a+3d\}$$

d: “common difference”

Meaning recursive definition:

$$a_1 = a - 1^{th}\ term$$
$$a_{n+1} = a_n + d, n = 1, 2, 3$$

The direct n^{th} term formula for an arithmetic sequence

If $\{a_n\}$ is an arithmetic sequence with 1^{th} term a, **and** common difference d

Then the end term is:

$$a_n = a + (n - 1) \cdot d$$

The n^{th}partial sum of an arithmetic sequence

If $\{a_n\}$ is an arithmetic sequence with 1^{th} term a, **and** common difference d

then its n^{th}partial sum is:

$$S_n = \sum_{i=1}^{n}(a + (n-1)d) = \sum_{i=1}^{n} a + \sum_{i=1}^{n} d(n-1) =$$

$$na + d\sum_{i=1}^{n}(n-1) = na + d \cdot \frac{n(n-1)}{2} =$$

$$\frac{n}{2}[2a + (n-1)d] = \frac{n}{2}[a + a + (n-1)d]$$

$$\frac{n}{2}[a + a_n]$$

$$\sum_{i=1}^{n}(n-1) =$$

$$\frac{n(n-1)}{2}$$

See below

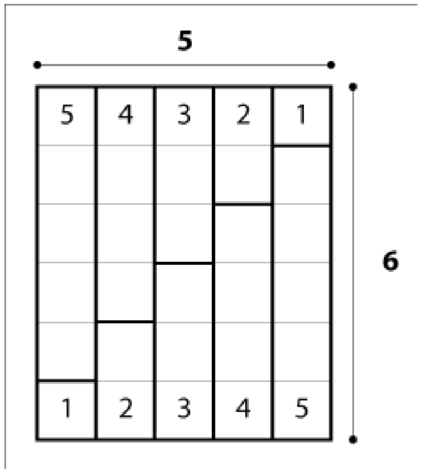
The sum of the first n natural numbers

Question: What is the sum of the first n natural numbers?

Answer:

$$S_n = \frac{n(n+1)}{2}$$

Geometric proof for $n = 5$



We see that we have a big rectangle with the its sides 5 and $5+1$. The rectangle has $2(1+2+3+4+5)$ squares inside. So $2(1+2+3+4+5) = 5(5+1)$ and $1 + 2 + 3 + 4 + 5 = \frac{5(5+1)}{2}$

http://www.9math.com/book/sum-first-n-natural-numbers

Geometric sequences

Definition: Let $a, r \in \mathbb{R}$, where $r \neq 0$. A geometric sequence has the standard form

$$\{a, ar, ar^2, ar^3\}$$

r: “common ratio”

Meaning recursive definition:

$$a_1 = a - 1^{th}\ term$$
$$a_{n+1} = a_n r, \quad for\ all\ n = 1, 2, 3$$

The direct n^{th} term formula for a geometric sequence

If $\{a_n\}$ is an geometric sequence with 1^{th} term a, **and** common ratio $r \neq 0$

Then the end term is:

$$a_n = ar^{n-1}$$

The n^{th}partial sum of a geometric sequence

If $\{a_n\}$ is an geometric sequence with 1^{th} term a, **and** common ratio $r \neq 0$

Then its n^{th} partial sum is:

$$S_n = a + ar + \cdots + ar^{n-1} = a\left(\frac{1 - r^n}{1 - r}\right)$$

For $n = 1, 2, 3$ and $r \neq 0, 1$

Proof: Assume $r \neq 0, 1$

$$S_n = a + \cancel{ar} + \cancel{ar^2} + \cdots + \cancel{ar^{n-2}} + \cancel{ar^{n-1}}$$

$$-$$

$$rS_n = \cancel{ar} + \cancel{ar^2} + \cdots + \cancel{ar^{n-2}} + \cancel{ar^{n-1}} + ar^n$$

$$S_n - rS_n = a - ar^n$$

$$S_n(1 - r) = a(1 - r^n)$$

$$\boxed{S_n = a\left(\frac{1 - r^n}{1 - r}\right)}$$

Observe: The n^{th} partial sums S_n of a sequence $\{a_k\}$

$$\{a_k\} = \{a_1, a_2, a_3, \dots, a_n, a_{n+1}, a_{n+2}\}$$

form their own sequence $\{S_n\}$:

$$S_n = \sum_{k=1}^{n} a_k, \quad n = 1,2, \dots$$

$$\{S_n\} = \{a_1; a_1 + a_2;\ a_1 + a_2 + a_3;\ a_1 + a_2 + a_3 + \cdots a_n\}$$

Example:

$$\{a_k\} = \{1,2,3,4,5,6,7,\dots\}$$

$$S_3 = \sum_{k=1}^{3} a_k = 1 + 2 + 3 = 6$$

$$\{S_5\} = \{1, 1+2, 1+2+3, 1+2+3+4, 1+2+3+4+5\} = \{1,3,6,10,15\}$$

Series

The sum of the terms of a sequence is called a **series**.

Given an infinite sequence of numbers $\{ a_n \}$, a **series** is informally the result of adding all those terms together: $a_1 + a_2 + a_3 + \cdots$

These can be written more compactly using the summation symbol $\sum$.The **index of summation**, k takes consecutive integer values from the lower limit, 1 to the **upper limit**, n. The term a_k is a **general term**.

Finite series	**Infinite series**
$\sum_{k=1}^{n} a_k$	$\sum_{k=1}^{\infty} a_k$

A **finite series** is a summation of a finite number of terms. An **infinite series** has an infinite number of terms and an upper limit of infinity.

Convergence of infinite series

If the sequence $\{S_n\}$ of partial sums **converges** to some real number L i.e. a limit

$$\lim_{n\to\infty} S_n = L$$

exists,

then the series is said to converge to L.

In this case we can write:

$$\sum_{k=1}^{\infty} a_k = \lim_{n\to\infty} S_n = L$$

L is also called the **sum of the series**. In general, we say that an **infinite series** has a **sum** if the **partial sums** form a sequence that has a real **limit**.

Limit of sequence of partial sums - Convergence of an infinite series

If

the limit of sequence of partial sums exists and is finite

then the sequence is called **convergent**

and the series is also called **convergent.**

Likewise, if

the limit of sequence of partial sums does not exist or is plus or minus infinity
then the sequence is called **divergent**

and the series is also called **divergent**.

Convergent and divergent infinite series

Example: Geometric series

Definition: The expression

$$a + ar + ar^2 + \cdots + ar^{k+1} + \cdots = \sum_{k=1}^{\infty} ar^{k-1}$$

with 1^{th} term a + common ratio $r \neq 0$ is called an **infinite geometric series**.

Theorem:

If $|r| < 1$, then the infinite geometric series $a + ar + ar^2 + \cdots + ar^{k+1} + \cdots$ has a finite sum for any constant a

$$\sum_{k=1}^{\infty} ar^{k-1} = \frac{a}{1-r}$$

Warning: If $|r| \geq 1$, then the $\frac{a}{1-r}$ sum formula is false.

Proof idea: Recall the n^{th} partial sum

$$S_n = a \cdot \left(\frac{1 - r^n}{1 - r}\right) = \frac{a}{1 - r} - \frac{a \cdot r^n}{1 - r}$$

Of course, as $n \to \infty$ $S_n \to$ Series "sum"

Since here $|r| < 1$, experience suggests that as $n \to \infty$, $|r|^n \to 0$ hence that as $n \to \infty$,

$$S_n = \frac{a}{1 - r} - \underbrace{\frac{a \cdot r^n}{1 - r}}_{\to 0} \to \frac{a}{1 - r}$$

The series **converges.**

The geometric series **diverges** whenever $r \leq -1$ or $r \geq 1$:

Example: $r = 1; a = 1$

$$\sum_{k=0}^{\infty} (-1)^k = 1 - 1 + 1 - 1 + 1 \ldots$$

diverges because its sequence of partial sums is

$$S_0 = 1$$

$$S_1 = 1 - 1 = 0$$

$$S_2 = 1 - 1 + 1 = 1$$

$$S_3 = 1 - 1 + 1 - 1 = 0$$

And the sequence $\{1,0,1,0, \ldots\}$ diverges.

The Factorial Symbol !

Definition: For $n = 0, 1, 2, 3 \ldots$ define "n factorial" $n!$ to be

$$0! = 1 \text{ (convenience),}$$

$$n! = n(n-1)(n-2) \ldots 3 \cdot 2 \cdot 1$$

Examples:

$$0! = 1$$

$$1! = 1$$

$$2! = 2 \cdot 1 = 2$$

$$3! = 3 \cdot 2 \cdot 1 = 6$$

Rapid growth:

$$20! \approx 2.4 \cdot 10^{18}$$

$$50! \approx 3 \cdot 10^{64}$$

Fact:

$$n! = n \cdot (n-1)! = n(n-1)(n-2)!$$

Example:

$$8! = 8 \cdot 7! = 8 \cdot 7 \cdot 6!$$

Warning:

$$(n - j)! \neq n! - j!$$

$$(n \cdot j)! \neq (n!) \cdot (j!)$$

Examples:

$$(5 - 2)! \neq 5! - 2!$$

$$3! = 6 \neq 120 - 2 = 118$$

$$(2 \cdot 3)! \neq 2! \cdot 3!$$

$$(6)! = 720 \neq 2! \cdot 3! = 2 \cdot 6 = 12$$

Exercises:

1.

$$\frac{6!}{3!} = \frac{6 \cdot 5 \cdot 4 \cdot (3!)}{3!} = 6 \cdot 5 \cdot 4 = 120$$

2.

$$7! \cdot 0! = 7!$$

Infinite convergent series: Examples

Many so-called **elementary functions** can be defined by series.

The exponential function e^x may be defined by the following power series:

$$e^x = \sum_{k=0}^{\infty} \frac{x^k}{k!} = 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24} + \cdots, \quad x \in \mathbb{R}$$

Cosine function

$$cos x = \sum_{k=0}^{\infty} (-1)^k \frac{x^{2k}}{(2k)!} = 1 - \frac{x^2}{2} + \frac{x^4}{24} \pm \cdots, \quad x \in \mathbb{R}$$

Statistics: Cumulative Distributions of Probability, Discrete Variable

$$P_k = \sum_{l=0}^{k} p_l \, , k = 0,1,2, \ldots$$

$$\lim_{k \to \infty} P_k = 1$$