Exercises Part 8 with Solutions

1. Find all local extrema of the function

$$f(x,y) = 2x^2 + 4xy - 3y^2 - 20x + 10y + 4$$

$$\frac{\partial f}{\partial x}(x,y) = 4x + 4y - 20$$

$$\frac{\partial f}{\partial y}(x,y) = 4x - 6y + 10$$

$$\Downarrow$$

$$\begin{cases} 4x + 4y - 20 = 0 \\ 4x - 6y + 10 = 0 \end{cases}$$

$$- \begin{array}{l} \cancel{4x} + 4y - 20 = 0 \\ \cancel{4x} - 6y + 10 = 0 \end{array}$$

$$\Downarrow$$

$$10y - 30 = 0$$

$$y = 3 \ and \ x = 2$$

Hessian matrix:

$$H(f) = \begin{pmatrix} \frac{\partial^2 f}{\partial x^2} & \frac{\partial^2 f}{\partial x \partial y} \\ \frac{\partial^2 f}{\partial x \partial y} & \frac{\partial^2 f}{\partial y^2} \end{pmatrix} = \begin{pmatrix} 4 & 4 \\ 4 & -6 \end{pmatrix}$$

$$D = \frac{\partial^2 f}{\partial x^2}\frac{\partial^2 f}{\partial y^2} - \left(\frac{\partial^2 f}{\partial x \partial y}\right)^2 = 4(-6) - 4 \cdot 4 = -40 < 0 \; saddle \quad \Rightarrow \quad \boxed{(2, 3, -1)}$$

$$f(2,3) = 2 \cdot 2^2 + 4 \cdot 2 \cdot 3 - 3 \cdot 3^2 - 20 \cdot 2 + 10 \cdot 3 + 4 = -1$$

2. Find the indefinite integral of the following functions.

a)

$$f(x) = (2x+1)(4x^3+2) + cos\ (x)$$

$$f(x) = 8x^4 + 4x^3 + 4x + 2 + cos\ (x)$$

$$F(x) = \frac{8}{5}x^5 + x^4 + 2x^2 + 2x + sinx + C$$

b)

$$f(x) = e^{2x+1} + \frac{4}{2x+3}$$

$$Substitutions: \begin{bmatrix} u = 2x+1; & \frac{du}{dx} = 2; dx = \frac{du}{2} \\ v = 2x+3; & \frac{dv}{dx} = 2; dx = \frac{dv}{2} \end{bmatrix}$$

$$\int e^{2x+1} dx = \int \frac{1}{2} e^{u} du = \frac{1}{2} e^{u} = \frac{1}{2} e^{2x+1} + C$$

$$\int \frac{4}{2x+3} dx = \int \frac{1}{2} \cdot \frac{4}{v} dv = 2\, ln\, v = 2\, ln(2x+3) + C$$

$$F(x) = \frac{1}{2} e^{2x+1} + 2\, ln(2x+3) + C$$

c)

$$f(x) = \left(x + \frac{1}{3}\right) sin\ x$$

Integration by Parts:

$$\int u'(x) \cdot v(x) dx = uv - \int u(x) \cdot v'(x)$$

$$\begin{bmatrix} u'(x) = sinx\,; u(x) = -cosx \\ v(x) = \left(x + \frac{1}{3}\right);\ v'(x) = 1 \end{bmatrix}$$

$$F(x) = \int sinx \cdot \left(x + \frac{1}{3}\right) dx = -cosx \cdot \left(x + \frac{1}{3}\right) - \int (-cosx \cdot 1) dx =$$
$$= -cosx \cdot \left(x + \frac{1}{3}\right) - (-sinx) = sinx - cosx\left(x + \frac{1}{3}\right) + C$$

3. Find the definite integrals:

a)

$$\int_{1}^{3} 2^{2x+1}\, dx$$

The Indefinite Integration by Substitution

$$Substitution: \left[u = 2x + 1;\ \frac{du}{dx} = 2; dx = \frac{du}{2}\right]$$

$$\int 2^{2x+1} dx = \int \frac{1}{2} 2^{u}\, du = \frac{2^{u}}{2ln2} = \frac{2^{2x+1}}{2ln2}$$

$$\int_{1}^{3} 2^{2x+1}\, dx = \left[\frac{1}{2ln2} 2^{2x+1}\right]_{1}^{3} = \frac{1}{2ln2}(2^{7} - 2^{3}) = \frac{1}{ln2} \cdot 60 \approx 86{,}56$$

$$\int_{1}^{3} 2^{2x+1}\, dx$$

The definite Integration by Substitution

$$\begin{bmatrix} u = 2x + 1 \\ u(1) = 3 \\ u(3) = 7 \end{bmatrix}$$

$$\int_{1}^{3} 2^{2x+1}\, dx = \left[\frac{1}{2ln2} 2^{u}\right]_{3}^{7} = \frac{1}{2ln2}\left(2^{7} - 2^{3}\right) \approx 86{,}56$$

b)

$$\int_{-4}^{-1} 3x^2 \cdot e^x \, dx$$

Integration by Parts:

$$\int u'(x) \cdot v(x) dx = uv - \int u(x) \cdot v'(x)$$

$$\begin{bmatrix} u'(x) = e^x \,; u(x) = e^x \\ v(x) = 3x^2; \; v'(x) = 6x \end{bmatrix}$$

$$\int 3x^2 \cdot e^x \, dx = e^x \cdot 3x^2 - \int e^x \cdot 6x \, dx = e^x \cdot 3x^2 - (e^x \cdot 6x - e^x \cdot 6) =$$
$$= 3e^x(x^2 - 2x + 2)$$

$$\int_{-4}^{-1} 3x^2 \cdot e^x \, dx = [3e^x(x^2 - 2x + 2)]_{-4}^{-1} =$$
$$= 3e^{(-1)}((-1)^2 - 2(-1) + 2) - 3e^{(-4)}((-4)^2 - 2(-4) + 2)) \approx 4{,}09$$

4. Compute the total area between the function

$$f(x) = \frac{1}{x} lnx$$

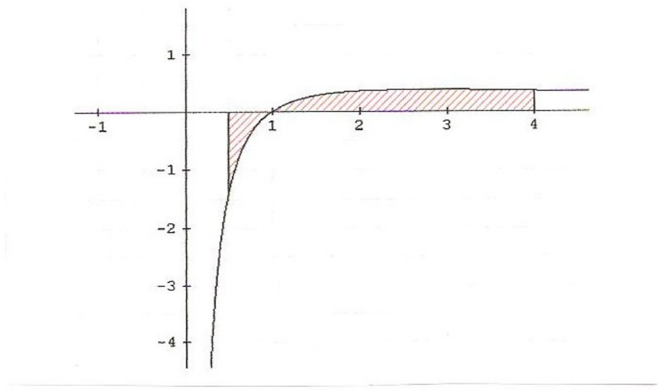
the x-axis and the lines $x_1 = 0{,}5$ and $x_2 = 4$

$$f(x) = \frac{1}{x} lnx$$

$$x_1 = 0{,}5 \; and \; x_2 = 4$$

The roots of the function:

$$\frac{1}{x} lnx = 0; lnx = 0 \Longrightarrow x = 1$$



$$f(x) = \frac{1}{x} lnx$$

The Indefinite Integration by Substitution

$$\left[\begin{array}{c} u = lnx;\ \frac{du}{dx} = \frac{1}{x};\ dx = xdu \\ \int \frac{1}{x} lnx = \int \frac{1}{\cancel{x}} u\cancel{x} du = \int udu = \frac{u^2}{2} = \frac{(lnx)^2}{2} \end{array}\right]$$

$$Area = \left|\int_{0,5}^{1} \frac{1}{x} lnxdx\right| + \int_{1}^{4} \frac{1}{x} lnx\ dx = \left|\left[\frac{(lnx)^2}{2}\right]_{0,5}^{1}\right| + \left[\frac{(lnx)^2}{2}\right]_{1}^{4} =$$

$$= \left|\frac{1}{2}(\ln (1))^2 - \frac{1}{2}(\ln (0,5))^2\right| + \frac{1}{2}(\ln (4))^2 - \frac{1}{2}(\ln(1))^2$$

$$= \left|0 - \frac{1}{2}(\ln(0,5))^2\right| + \frac{1}{2}(\ln(4))^2 - 0 \approx 1,20$$

5. Compute the total area between the following curves $f(x)$ and $g(x)$:

$$f(x) = 2x + 1; \quad g(x) = (x-1)^2$$

Intersections:

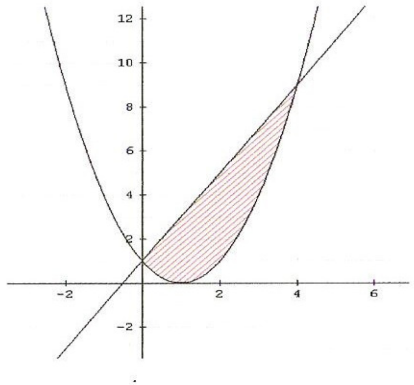
$$2x + 1 = (x-1)^2$$

$$2x + 1 = x^2 - 2x + 1$$

$$x^2 - 4x = 0$$

$$x(x-4) = 0$$

$$x_1 = 0;\ x_2 = 4$$



Select some $x \in (0,4)$, e,g, $x = 1$: $f(1) = 2 + 1 = 3 > g(1) = (1-1)^2 = 0$;

$$\Downarrow$$

$$f(x) = 2x + 1: ceiling;\ g(x) = (x-1)^2: floor$$

$$Area = \int_0^4 (2x + 1 - (x-1)^2)dx =$$

$$= \int_0^4 \left(2x + 1 - (x^2 - 2x + 1)\right)dx = \int_0^4 (-x^2 + 4x)dx = \left[-\frac{1}{3}x^3 + 2x^2\right]_0^4 =$$

$$= -\frac{1}{3} \cdot 4^3 + 2 \cdot 4^2 + 0 - 0 = 10\frac{2}{3}$$