Exercise Part 7 with Solutions

1. Find the limits of the following functions with L'Hopital rule:

a)

$$\lim_{x \to 1} \frac{5x^4 - 4x^2 - 1}{-9x^3 - x + 10} = \lim_{x \to 1} \frac{20x^3 - 8x}{-27x^2 - 1} = \frac{20 - 8}{-27 - 1} = -\frac{3}{7}$$

b)

$$\lim_{x\to\infty}\left[\frac{e^{3x}}{x^3}\right] = \lim_{x\to\infty}\left[\frac{3e^{3x}}{3x^2}\right] = \lim_{x\to\infty}\left[\frac{9e^{3x}}{6x}\right] = \lim_{x\to\infty}\left[\frac{27e^{3x}}{6}\right] = \infty$$

2. Find all local extrema of the functions

a)

$$f(x) = x^2 - 4x + 5$$

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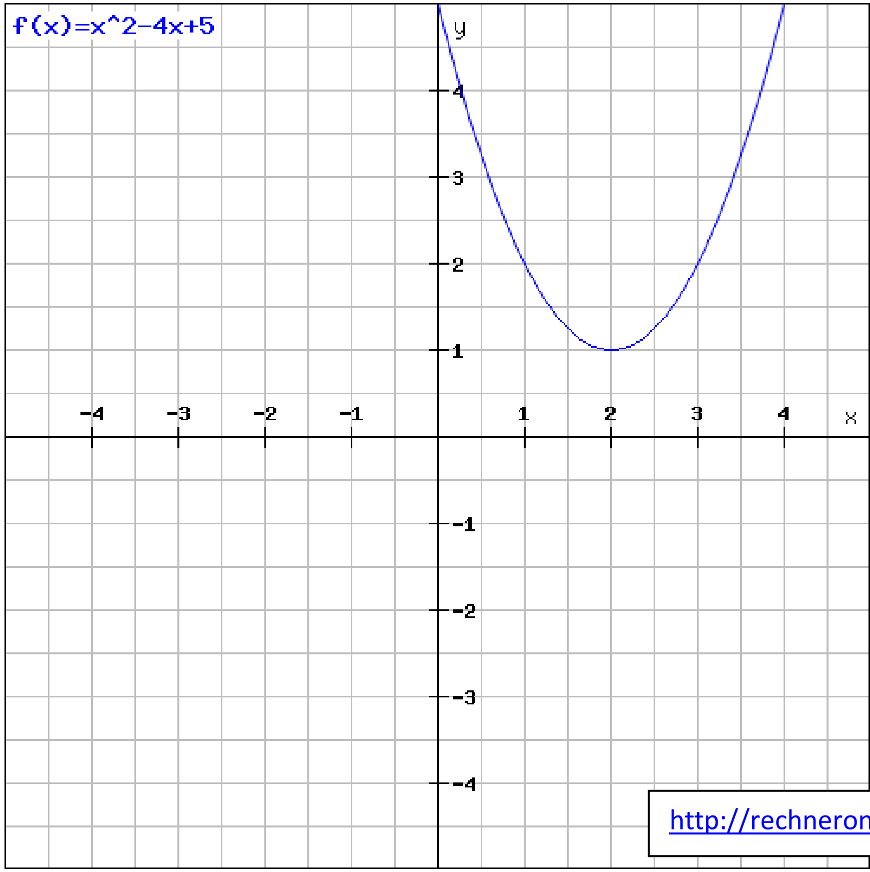
Solution:

$$f'(x) = 2x - 4 = 0 \qquad x = 2 \qquad f''(x) = 2 > 0$$

$x = 2$: local minimum



http://rechneronline.de/funktionsgraphen/

b)

$$f(x) = \frac{2x^2 + 8}{x}$$

$$f(x) = \frac{2x^2 + 8}{x}$$

Solution

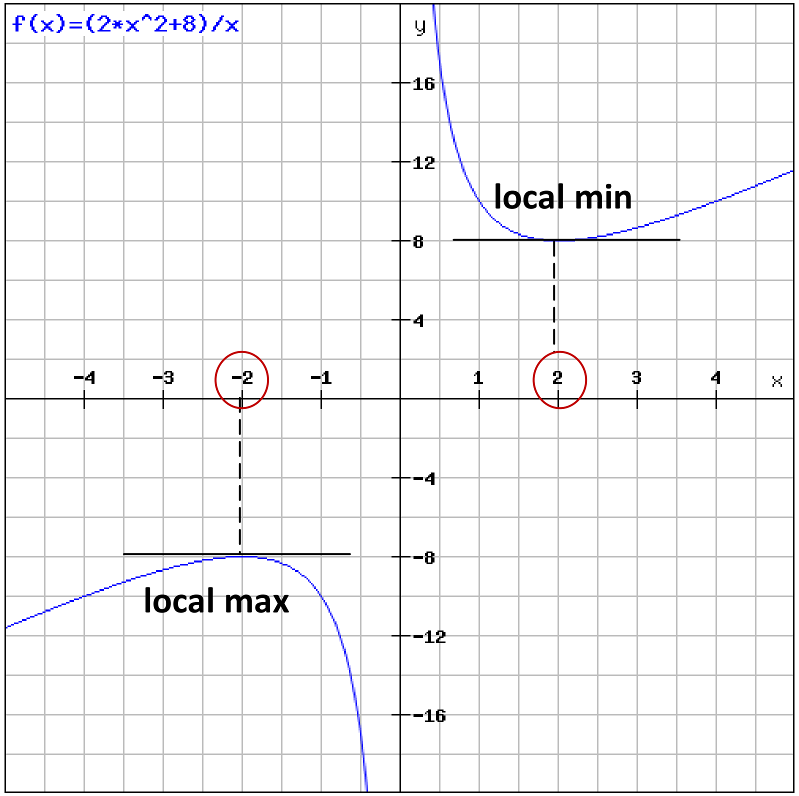
$$f'(x) = \frac{4x \cdot x - 1 \cdot (2x^2 + 8)}{x^2} = \frac{4x^2 - 2x^2 - 8}{x^2} = \frac{2x^2 - 8}{x^2}$$

$$2x^2 - 8 = 0 \qquad x_1 = 2, x_2 = -2$$

$$f''(x) = \frac{4x \cdot x^2 - 2x(2x^2 - 8)}{x^4} = \frac{4x^3 - 4x^3 + 16x}{x^4} = \frac{16}{x^3}$$

$x = 2$ $\qquad f''(x) = \frac{16 \cdot 2}{2^4} = 2 > 0$ $\qquad$ local minimum

$x = -2$ $\qquad f''(x) = \frac{16 \cdot (-2)}{(-2)^4} = -2 < 0$ $\qquad$ local maximum


f(x)=(2*x^2+8)/x
y
16
12
8
4
-4
-3
-2
-1
1
2
3
4
x
-4
-8
-12
-16
local min
local max

4. Find where the following functions is increasing/decreasing, concave up/down and all inflection points

$$f(x) = x^3 - 4x^2 + 2x + 4$$

Solution: Increasing/decreasing

$$f(x) = x^3 - 4x^2 + 2x + 4$$

$$f'(x) = 3x^2 - 8x + 2$$

$$3x^2 - 8x + 2 = 0$$

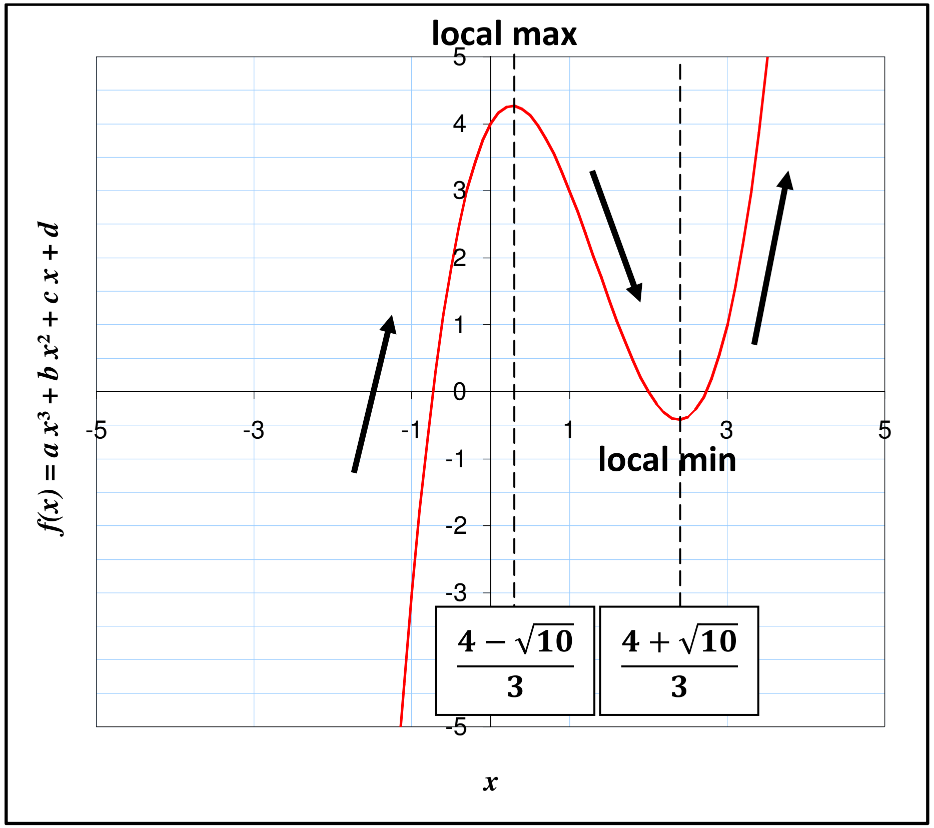
$$x_{1,2} = \frac{8 \pm \sqrt{64 - 4 \cdot 3 \cdot 2}}{6} = \frac{4 \pm \sqrt{10}}{3}$$

$$f'(x) = \left(x - \frac{4 - \sqrt{10}}{3}\right)\left(x - \frac{4 + \sqrt{10}}{3}\right)$$

$$-\infty < x < \frac{4 - \sqrt{10}}{3} : f'(x) > 0: \ increasing$$

$$\frac{4 - \sqrt{10}}{3} < x < \frac{4 + \sqrt{10}}{3} : \ f'(x) < 0: decreasing$$

$$x > \frac{4 + \sqrt{10}}{3} : f'(x) > 0 \ increasing$$


local max
local min
$\frac{4-\sqrt{10}}{3}$
$\frac{4+\sqrt{10}}{3}$
$f(x) = a\,x^3 + b\,x^2 + c\,x + d$
x
5
4
3
2
1
0
-1
-2
-3
-5
-5
-3
-1
1
3
5

Maximum/Minimum

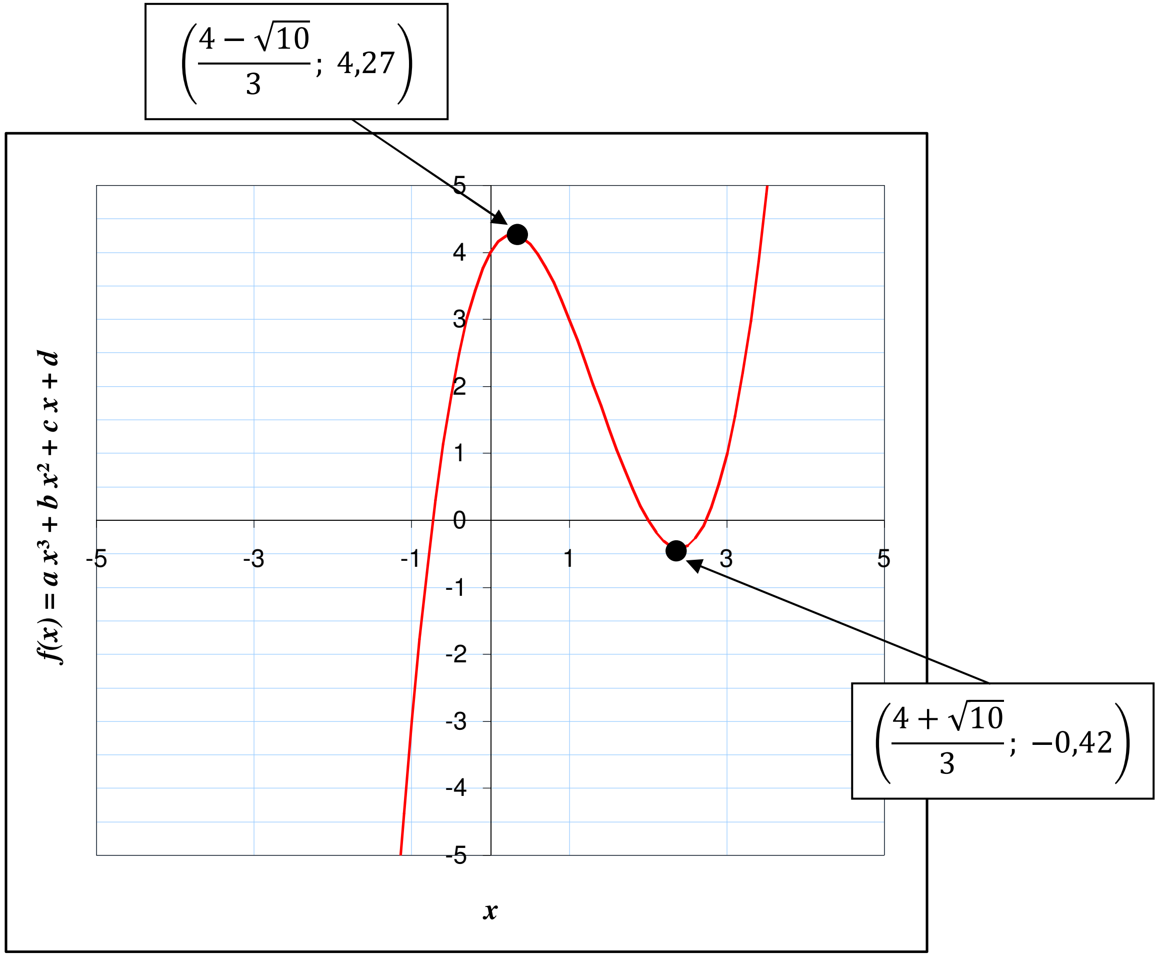
$$f''(x) = 6x - 8$$

$$x_1 = \frac{4-\sqrt{10}}{3} \qquad f''(x_1) = 6 \cdot \frac{4-\sqrt{10}}{3} - 8 = -2\sqrt{10} < 0 - maximum$$

$$f(x_1) = \left(\frac{4-\sqrt{10}}{3}\right)^3 - 4\left(\frac{4-\sqrt{10}}{3}\right)^2 + 2\left(\frac{4-\sqrt{10}}{3}\right) + 4 = \boxed{4{,}27}$$

$$x_2 = \frac{4+\sqrt{10}}{3} \qquad f''(x_2) = 6 \cdot \frac{4+\sqrt{10}}{3} - 8 = 2\sqrt{10} > 0 - minimum$$

$$f(x_2) = \left(\frac{4+\sqrt{10}}{3}\right)^3 - 4\left(\frac{4+\sqrt{10}}{3}\right)^2 + 2\left(\frac{4+\sqrt{10}}{3}\right) + 4 = \boxed{-0{,}42}$$



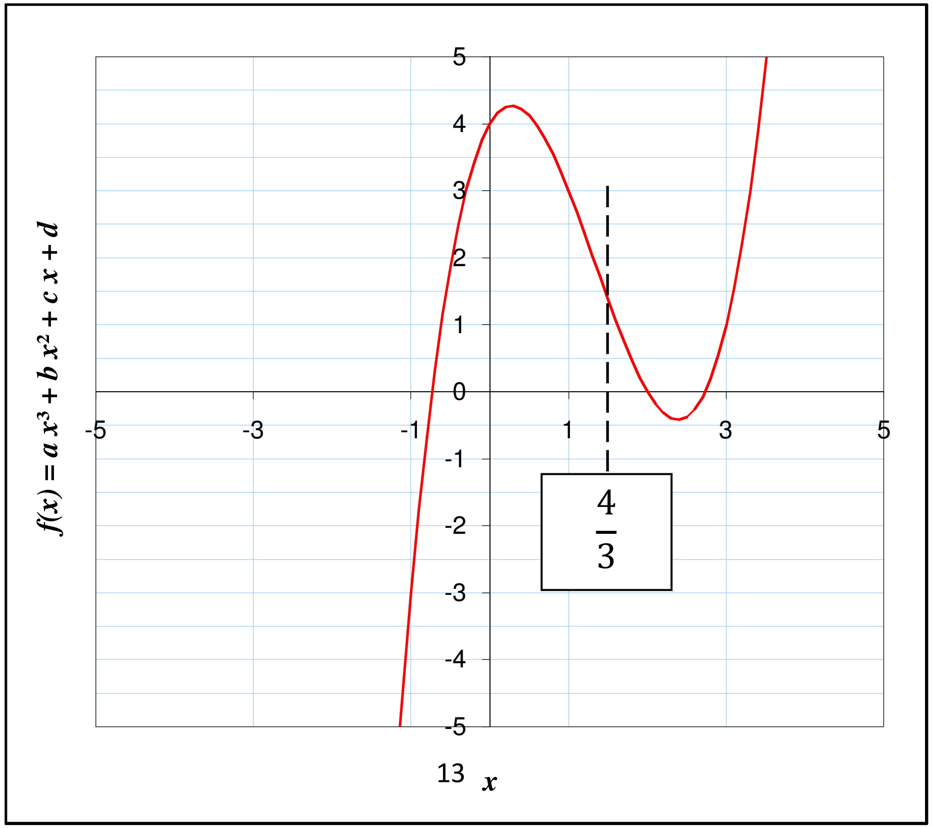
$\left(\frac{4-\sqrt{10}}{3};\ 4{,}27\right)$
$\left(\frac{4+\sqrt{10}}{3};\ -0{,}42\right)$
$f(x) = a\,x^3 + b\,x^2 + c\,x + d$
x
5
4
3
2
1
0
-1
-2
-3
-4
-5
-5
-3
-1
1
3
5

Concave up/down

$f''(x) < 0$: $\quad 6x - 8 < 0 \quad x < \frac{4}{3}$ $\qquad$ concave down

$f''(x) > 0$: $\quad 6x - 8 > 0 \quad x > \frac{4}{3}$ $\qquad$ concave up

$x = \frac{4}{3}$ $\qquad$ inflection point



4. Find the equation of a tangent plane to the function:

$$f(x, y) = 2x + \ln(x^2 + y)$$

at the point $x_0 = 1, y_0 = 0$

Funktion: $f(x,y) = 2x + \ln(x^2 + y)\,;\; x_0 = 1, y_0 = 0$

Solution:

The equation of a tangent plane: $T_{(x_0,y_0)}(x,y) = f(x_0,y_0) + \nabla f_{(x_0,y_0)} \begin{pmatrix} x - x_0 \\ y - y_0 \end{pmatrix}$

$$\boxed{f(x_0,y_0)} = f(1{,}0) = 2 + \ln(1^2 + 0) = 2 + ln1 = \boxed{2}$$

$$\nabla f = \begin{pmatrix} 2 + \dfrac{2x}{x^2 + y} \\ \dfrac{1}{x^2 + y} \end{pmatrix}$$

$$\boxed{\nabla f_{(x_0,y_0)}} = \nabla f_{(1,0)} = \begin{pmatrix} 2 + \dfrac{2 \cdot 1}{(1^2 + 0)} \\ \dfrac{1}{(1^2 + 0)} \end{pmatrix} = \begin{pmatrix} 2 + 2 \\ \dfrac{1}{1} \end{pmatrix} = \boxed{\begin{pmatrix} 4 \\ 1 \end{pmatrix}}$$

$$\boxed{T_{(x_0,y_0)}(x,y) = f(x_0,y_0) + \nabla f_{(x_0,y_0)} \begin{pmatrix} x - x_0 \\ y - y_0 \end{pmatrix}}$$

$$f(x_0,y_0) = 2$$

$$\nabla f_{(x_0,y_0)} = \begin{pmatrix} 4 \\ 1 \end{pmatrix}$$

$$T_{(1,0)}(x,y) = f(1,0) + \nabla f_{(1,0)} \begin{pmatrix} x - 1 \\ y - 0 \end{pmatrix} =$$

$$2 + \begin{pmatrix} 4 \\ 1 \end{pmatrix} \begin{pmatrix} x - 1 \\ y - 0 \end{pmatrix} =$$

$$2 + 4(x-1) + 1(y-0) = 2 + 4x - 4 + y - 0 = 4x + y - 2$$