Exercises (Sets, linear algebra and calculus)

Task 1 *Sets*

The following sets are given:

$A = \{1; 2; 3\}$, $B = \{ x \in \mathbb{N} \mid x \geq 1 \wedge x < 10 \wedge x \text{ odd} \}$, $C = \{ 0; 1\}$.

Describe the following sets by enumerating their elements:

(a) B

(b) the union $A \cup B$

(c) the intersection $A \cap B$

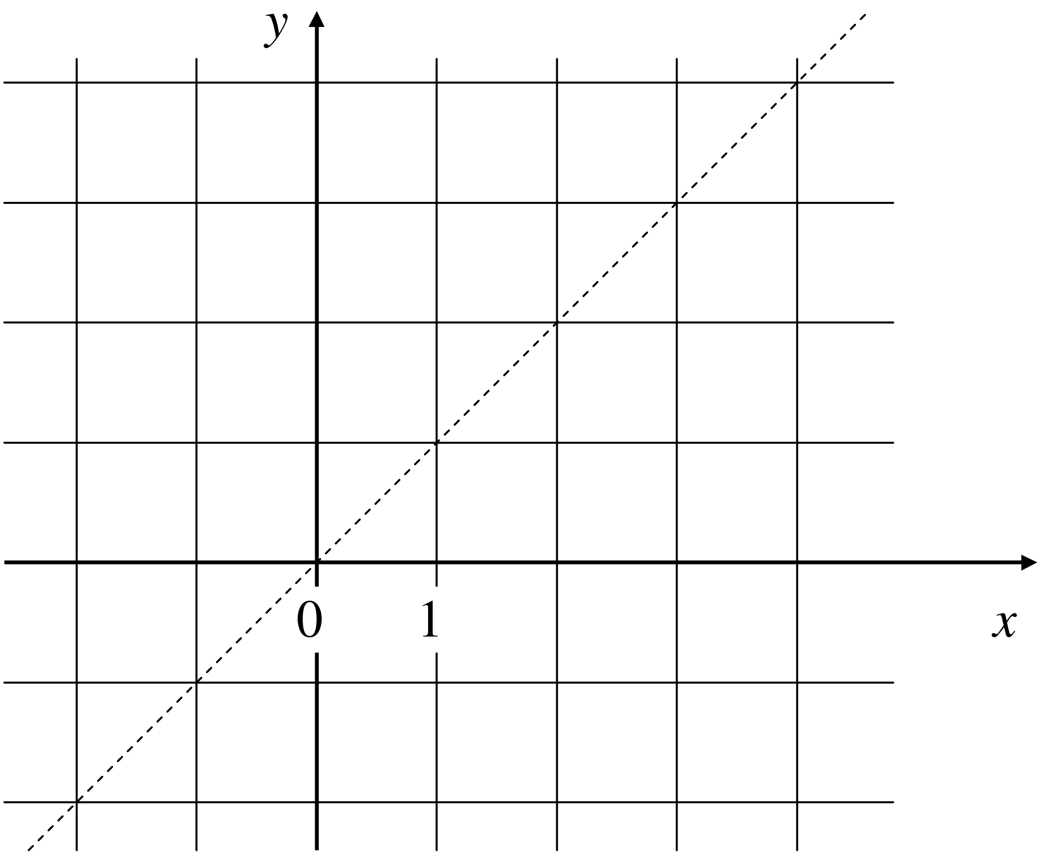
(d) the power set $\mathcal{P}(A)$

(e) the Cartesian product $\mathrm{A} \times \mathrm{C}$

(f) the Cartesian product $\mathrm{C} \times \mathrm{C} \times \mathrm{C}$.

Task 2 *Vectors and linear mappings in the plane*

Let f: $\mathrm{R}^2 \to \mathrm{R}^2$ be the mirror transformation at the principal bisector, i.e., at the line $x = y$ (see Figure).



(a) Draw the vectors $\vec{e}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, $\vec{e}_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$, $\vec{a} = \begin{pmatrix} 3 \\ 1 \end{pmatrix}$, $\vec{b} = \begin{pmatrix} -2 \\ -2 \end{pmatrix}$

and their images $f(\vec{e}_1)$, $f(\vec{e}_2)$, $f(\vec{a})$, $f(\vec{b})$ into the above figure.
If some of these vectors coincide, write down the equality.

(b) What is the matrix A which describes f
(i.e., for which $A \cdot \vec{x} = f(\vec{x})$ holds for all $\vec{x} \in \mathrm{R}^2$) ?
(Hint: f is a linear mapping; you need not prove this.)

(c) Calculate $A \cdot A$.

(d) How can the result of (c) be interpreted geometrically?

Task 3 *Vectors in space*

Given is a triangle with the corners $A = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$, $B = \begin{pmatrix} 4 \\ 0 \\ 3 \end{pmatrix}$, $C = \begin{pmatrix} 4 \\ 1 \\ 2 \end{pmatrix}$.

Please calculate:

(a) the vectors $\overrightarrow{AB}$, $\overrightarrow{BC}$ and $\overrightarrow{AC}$, which each connect two of the corners,

(b) the lengths of the three sides of the triangle,

(c) the cosine of the inner angle of the triangle at the corner A,

(d) the area of the triangle.

(Give also all intermediate calculations! Roots, like, e.g., $\sqrt{42}$, need not be calculated numerically.)

Task 4 *Matrices and determinants*

Given are the matrices $A = \begin{pmatrix} 1 & 4 & 3 \\ 0 & 2 & 1 \\ 0 & 7 & 4 \end{pmatrix}$ and $B = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 5 \end{pmatrix}$.

(a) Calculate the determinants of A and B.

(b) Calculate the matrix $A \cdot B$.

(c) Calculate the inverse matrix A^{-1} .

Task 5 *Solvability of a system of linear equations*

How many solutions has the following system of linear equations?

$$\begin{array}{rrrrl} x_1 & +\frac{1}{2}x_2 & +17x_3 & -3x_4 & =0 \\ & 2x_2 & +5x_3 & +22x_4 & =5 \\ & & x_3 & +3x_4 & =4 \\ & & 2x_3 & +6x_4 & =8 \end{array}$$

Give a precise reason for your result.

Task 6 *Systems of linear equations, determinant*

Given is the matrix $A = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 0 & 1 & 1 \end{pmatrix}$ and the vector $\vec{b} = \begin{pmatrix} 200 \\ 320 \\ 80 \end{pmatrix}$.

Furthermore, $\vec{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$.

(a) $A \cdot \vec{x} = \vec{b}$ describes in a concise way a system of 3 linear equations with 3 unknowns. Write down these three equations.

(b) Find a solution vector $\vec{x}$ of the system.

(c) Give a reason (without calculation!) why the determinant of A cannot be 0.

Task 7 *Real-valued functions of one variable; limits; local extrema*

Given are the functions

$$f(x) = 5x^3 - \frac{15}{2}x^2 - 30x + 50$$
$$g(x) = 2 - \frac{1}{4}x^3$$
$$h(x) = 1 - 2x^2$$

(a) Determine the limit values

$$\lim_{x \to \infty} \frac{f(x)}{g(x)}$$

$$\lim_{x \to 0} \frac{f(x)}{g(x)}$$

$$\lim_{x \to 2} \frac{f(x)}{g(x)}$$

(b) Determine the positions (x values) of the local extrema of f: where does this function reach a minimum, where a maximum?

(c) Draw the function h in the Cartesian coordinate system (approximately).
Prove that h is not injective.

(d) Calculate $h(g(x))$. (Simplify the term as far as possible.)

Task 8 *Extremal points of functions of two variables*

Given is the function $f(x, y) = 4x^2y + 2xy - 3y^2 + 5$.

(a) Calculate the following partial derivatives: f_x, f_y, f_{xx}, f_{xy}, f_{yy} .

(b) Calculate all critical points (x, y) of f (i.e., all points where f_x and f_y are both 0).

(c) Indicate for each critical point if it is a saddle point or a local extremal point of f, and in the latter case, if it is a maximum or a minimum.

Task 9 *Integration*

Calculate the values of the following integrals:

(a) $$\int_0^2 (x^3 - x)\, dx$$

(b) $$\int_0^{\pi} \sin x \, dx$$