

Exercise 7

1. Find limits of following functions with L'Hopital rule:

a)

$$\lim_{x \rightarrow 1} \frac{5x^4 - 4x^2 - 1}{-9x^3 - x + 10} = \lim_{x \rightarrow 1} \frac{20x^3 - 8x}{-27x^2 - 1} = \frac{20 - 8}{-27 - 1} = -\frac{3}{7}$$

b)

$$\lim_{x \rightarrow \infty} \left[\frac{e^{3x}}{x^3} \right] = \lim_{x \rightarrow \infty} \left[\frac{3e^{3x}}{3x^2} \right] = \lim_{x \rightarrow \infty} \left[\frac{9e^{3x}}{6x} \right] = \lim_{x \rightarrow \infty} \left[\frac{27e^{3x}}{6} \right] = \infty$$

2. Find all local extrema of the functions

a)

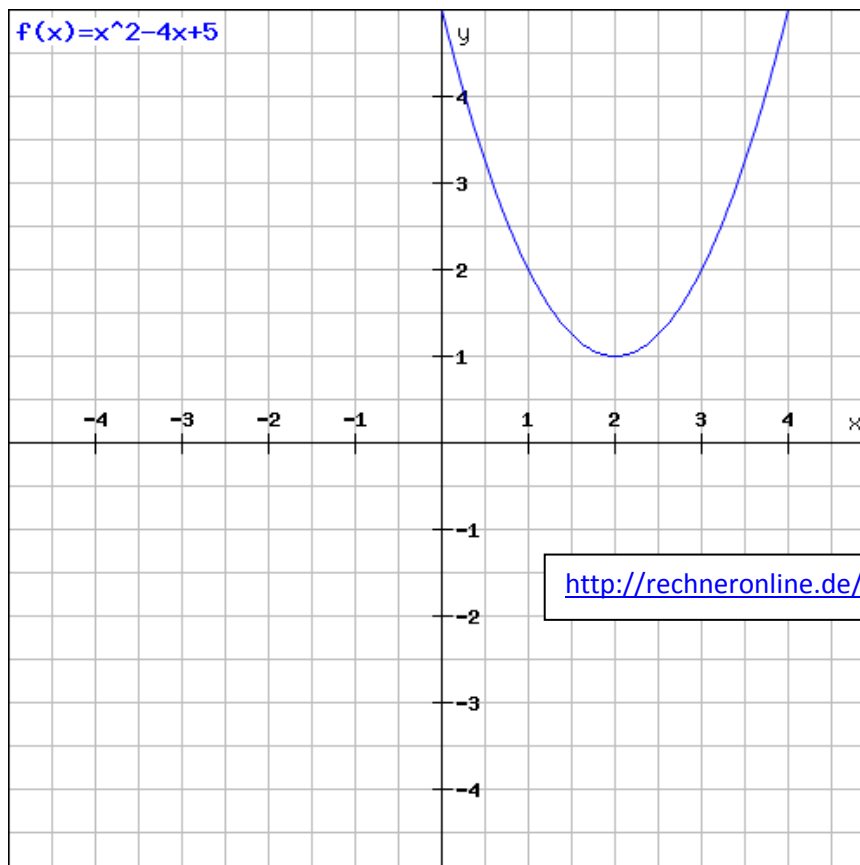
$$f(x) = x^2 - 4x + 5$$

$$f'(x) = 2x - 4 = 0$$

$$x = 2$$

$$f''(x) = 2 > 0$$

$x = 2$: local minimum



b)

$$f(x) = \frac{2x^2 + 8}{x}$$

$$f'(x) = \frac{4x \cdot x - 1 \cdot (2x^2 + 8)}{x^2} = \frac{4x^2 - 2x^2 - 8}{x^2} = \frac{2x^2 - 8}{x^2}$$

$$2x^2 - 8 = 0$$

$$x_1 = 2, x_2 = -2$$

$$f''(x) = \frac{4x \cdot x^2 - 2x(2x^2 - 8)}{x^4} = \frac{4x^3 - 4x^3 + 16x}{x^4} = \frac{16}{x^3}$$

$x = 2$:

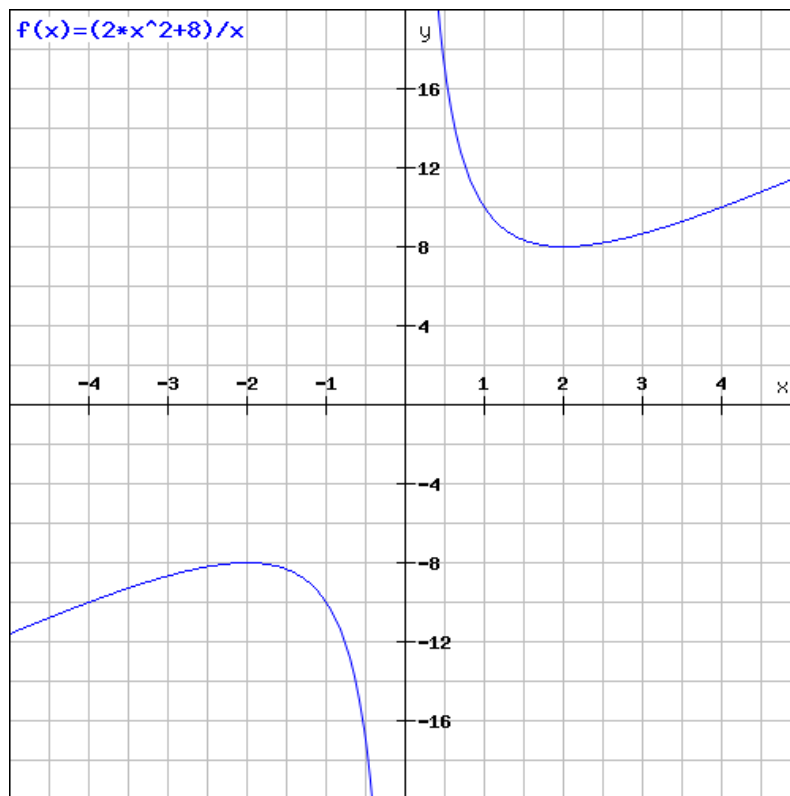
$$f''(x) = \frac{16 \cdot 2}{2^4} = 2 > 0$$

local minimum

$x = -2$:

$$f''(x) = \frac{16 \cdot (-2)}{(-2)^4} = -2 < 0$$

local maximum



3. Find where the following functions is increasing/decreasing, concave up/down and all inflection points

$$f(x) = x^3 - 4x^2 + 2x + 4$$

$$f'(x) = 3x^2 - 8x + 2$$

$$3x^2 - 8x + 2 = 0$$

$$x_{1,2} = \frac{8 \pm \sqrt{64 - 4 \cdot 3 \cdot 2}}{6} = \frac{4 \pm \sqrt{10}}{3}$$

$$f'(x) = \left(x - \frac{4 - \sqrt{10}}{3}\right) \left(x - \frac{4 + \sqrt{10}}{3}\right)$$

$$-\infty < x < \frac{4 - \sqrt{10}}{3} : f'(x) > 0: \textit{increasing}$$

$$\frac{4 - \sqrt{10}}{3} < x < \frac{4 + \sqrt{10}}{3} : f'(x) < 0: \textit{decreasing}$$

$$x > \frac{4 + \sqrt{10}}{3} : f'(x) > 0 \textit{ increasing}$$

$$f''(x) = 6x - 8$$

$$x_1 = \frac{4 - \sqrt{10}}{3}; f''(x_1) = 6 \cdot \frac{4 - \sqrt{10}}{3} - 8 = -2\sqrt{10} < 0 - \text{maximum}$$

$$f(x_1) = \left(\frac{4 - \sqrt{10}}{3}\right)^3 - 4\left(\frac{4 - \sqrt{10}}{3}\right)^2 + 2\left(\frac{4 - \sqrt{10}}{3}\right) + 4 = 4,27$$

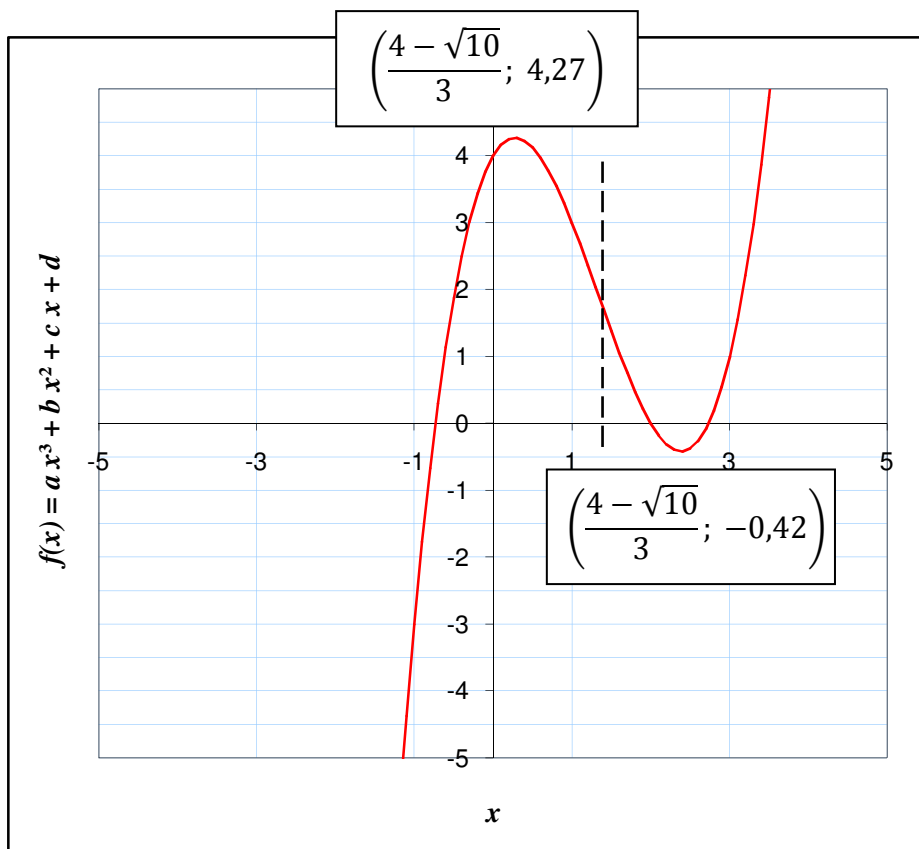
$$x_2 = \frac{4 + \sqrt{10}}{3}; f''(x_2) = 6 \cdot \frac{4 + \sqrt{10}}{3} - 8 = 2\sqrt{10} > 0 - \text{minimum}$$

$$f(x_2) = \left(\frac{4 + \sqrt{10}}{3}\right)^3 - 4\left(\frac{4 + \sqrt{10}}{3}\right)^2 + 2\left(\frac{4 + \sqrt{10}}{3}\right) + 4 = -0,42$$

$$f''(x) < 0: \quad 6x - 8 < 0 \quad x < \frac{4}{3} \quad \text{concave up}$$

$$f''(x) > 0: \quad 6x - 8 > 0 \quad x > \frac{4}{3} \quad \text{concave down}$$

$$x = \frac{4}{3} \quad \text{inflection point}$$



4. Find the equation of a tangent plane to:

$$f(x, y) = 2x + \ln(x^2 + y)$$

at the point $x_0 = 1, y_0 = 0$

$$\nabla f = \begin{pmatrix} 2 + \frac{2x}{x^2 + y} \\ \frac{1}{x^2 + y} \end{pmatrix}$$

$$T_{(1,0)}(x, y) = f(1,0) + \nabla f_{(1,0)} \begin{pmatrix} x - 1 \\ y - 0 \end{pmatrix}$$

$$f(1,0) = 2 + \ln(1^2 + 0) = 2 + \ln 1 = 2$$

$$\nabla f_{(1,0)} = \begin{pmatrix} 2 + \frac{2 \cdot 1}{(1^2 + 0)} \\ \frac{1}{(1^2 + 0)} \end{pmatrix} = \begin{pmatrix} 2 + 2 \\ \frac{1}{1} \end{pmatrix} = \begin{pmatrix} 4 \\ 1 \end{pmatrix}$$

$$T_{(1,0)}(x, y) = 2 + \begin{pmatrix} 4 \\ 1 \end{pmatrix} \begin{pmatrix} x - 1 \\ y - 0 \end{pmatrix} =$$

$$2 + 4(x - 1) + 1(y - 0) = 2 + 4x - 4 + y - 0 = 4x + y - 2$$

5. Find all inflection points of the function

$$f(x, y) = 2x^2 + 4xy - 3y^2 - 20x + 10y + 4$$

$$\frac{\partial f}{\partial x}(x, y) = 4x + 4y - 20$$

$$\frac{\partial f}{\partial y}(x, y) = 4x - 6y + 10$$

$$I) 4x + 4y - 20 = 0$$

$$II) 4x - 6y + 10 = 0$$

$$I - II) 10y - 30 = 0$$

$$y = 3 \text{ und } x = 2$$

$$\text{Hessian matrix: } \begin{pmatrix} 4 & 4 \\ 4 & -6 \end{pmatrix}$$

$$\frac{\partial^2 f}{\partial x^2} \frac{\partial^2 f}{\partial y^2} - \left(\frac{\partial^2 f}{\partial x \partial y} \right)^2 = 4(-6) - 4 \cdot 4 = -40 < 0 \text{ saddle}$$